TJP TOP TIPS

FOR

IGCSE ALGEBRA

IGCSE ALGEBRA

First, some important words; know what they mean (get someone to test you):

Expression – a fragment of algebra with no '=' sign.
$3x+5$ is an expression, and so is πr^2 .

Formula – tells you the connection between two or more quantities.
$A = \pi r^2$ is a formula giving the area of a circle in terms of its radius.

Equation – a mathematical statement with an '=' sign; it is only true for certain values.
$3x+5 = 14$ is an equation which is only true when $x = 3$.

Inequality – a mathematical statement with a '<', '>', '≤' or '≥' sign.
$3x+5 < 14$ is an inequality which is only true when $x < 3$.

Expand – get rid of brackets by multiplying out in full (opposite of Factorise).
a) Expand $3(2x-5y)$ Answer $6x-15y$
b) Expand $(x-5)(x+2)$ Answer $x^2-3x-10$

Factorise – put into brackets (opposite of Expand).
a) Factorise $6xy-4x^2$ Answer $2x(3y-2x)$
b) Factorise x^2+5x+4 Answer $(x+1)(x+4)$

Simplify – gather together any matching bits.
a) Simplify $3x+4y-5x+6y$ Answer $-2x+10y$
b) Simplify $4x^3y^2\times 3xy^4$ Answer $12x^4y^6$

Solve – work out what number(s) make an equation true.
a) Solve $3x-7 = 26$ Answer $3x = 33$ so $x = 11$
b) Solve $(x+2)(x-9) = 0$ Answer $x = -2$ or $x = 9$

Term – a 'bit' of an equation or an expression separated by + or – signs.
The second term of $3x-4y+5z$ is $-4y$.

Coefficient – the number part of a term.
The coefficient of x in $2x^2-7x+9$ is –7.

Linear – something with x in it (but no higher powers or roots).
$3x-5$ is a linear expression.

Quadratic – something with x² in it (but no higher powers or roots).
$3x^2+7x-8$ is a quadratic expression.

Function – a mathematical rule for changing an input number into an output number.
$f(x)=2x+1$ takes an input number, doubles it and adds 1 to give an output number.

Domain – the set of all numbers that can go into a function (domaIN).
The domain of $g(x)=\sqrt{x-4}$ is $x\geq 4$ because we can't square root a negative number.

Range – the set of all number that can come out of a function.
The range of $g(x)=\sqrt{x-4}$ is $g(x)\geq 0$ because the numbers coming out are 0 or above.

Inverse function – a mathematical rule that 'undoes' a given function (reverse flowchart).
$f^{-1}(x)=(x-1)/2$ is the inverse function of $f(x)=2x+1$.

Differentiate – find a formula for the gradient (the derivative or dy/dx) of a given curve.
If $y = 4x^5-2x^3+3x-5$, then $\frac{dy}{dx} = 20x^4-6x^2+3$.

Turning point (maximum or minimum) – a point on a curve having zero gradient.
The curve $y = x^2+3$ has a minimum at (0, 3) since $\frac{dy}{dx} = 2x = 0$ at $x = 0$.

IGCSE ALGEBRA

ALGEBRA BASICS

▷ **Algebra** is a branch of mathematics where letters are used instead of numbers. Why?
(i) we want a **formula** that **works for any value** of radius (let's say), so we call it r.
(ii) we **don't yet know the number** we want (we're solving an equation to find x).
(iii) we're plotting a graph where x & y are **always changing**, not fixed numbers.

You may not have thought about this much, but certain letters are used for certain things. **Don't bother learning this list** off by heart, but it might be useful as a reference.

$a, b, c,$ etc. are **numbers which are fixed** for a particular question (**constants**).
Write down the values of a, b and c where $ax^2+bx+c = 0$.

a, b, c are also the **unknown sides of a triangle**, whether right-angled or not.
Find the hypotenuse c using Pythagoras' Theorem $a^2+b^2 = c^2$.
A, B, C are the **angles** opposite sides a, b, c in a non right-angled triangle.

c is also the **y-intercept** of a **straight line**.
Find c if $y = 2x+c$ passes through the point (1, 8).

f, g, h are **functions**.
Find the inverse function of $f(x) = 4x-5$.

h can also be **height**.
Evaluate $\frac{1}{3}\pi r^2 h$ to find the volume of the cone.

l is usually **length**.
Curved surface area = $\pi r l$.

m can be the **gradient** of a **straight line**, or it can be **mass**.
Express in the form $y = mx+c$.

n is a variable, **unknown whole number**.
Find the 100th term of the sequence $t_n = n^2+1$.

r is usually **radius**.
Work out the value of πr^2.

s can be **distance**.
If $s = \frac{1}{2}(u+v)t$, rearrange this formula to make u the subject.

t is usually **time**.
Sketch the graph for $0 \le t \le 10$.

u, v can be **speed** or **velocity**.
Make m the subject in $I = mv-mu$.

w is usually **width**.
Express the perimeter in terms of l and w.

x, y, z are used for **co-ordinates** as well as for **unknown numbers in equations**.
Plot the line $y = 3x+5$; solve the equation $x^2+4x+3 = 0$.

CAPITAL LETTERS are often used for **unknown length-based quantities** such as:
A = area, C = circumference, L = length, P = perimeter, V = volume.

$\theta, \phi, \alpha, \beta$ are Greek letters (theta, phi, alpha, beta) used for **unknown angles**.
Find the value of angle θ, showing all your working.

IGCSE ALGEBRA

TIPS FOR WRITING ALGEBRA

- ▷ We write $2x$ instead of $2\times x$ to mean 'two lots of x' or $x+x$.
  We **leave out the times symbol** because it might get confused with x.

- ▷ We write x^2 instead of $x\times x$ or $x\,x$ to mean 'x times itself'.
  Using **indices** is clearer and quicker once we get to x^9 instead of $x\,x\,x\,x\,x\,x\,x\,x\,x\ldots$

- ▷ If you're multiplying a whole load of numbers and letters, remember:
  Put the number first, followed by the letters in alphabetical order.
  This makes it easier to see if two terms **match** (contain the same letters).
  So we'd write $3x^2yz$ rather than $z\,y\,x^2 3$.

- ▷ We hardly ever use the division symbol '÷'; instead we use a **division bar**, $\frac{x}{y}$.

- ▷ An '=' sign in algebra is not an instruction to work out the answer (like on a calculator).
  '=' means 'is the same as' or **'balances'**, not 'makes' or 'write the answer here'.

  Never write '=' between two things that aren't the same! $2+2=4+3=7$ is **wrong**!

- ▷ Use **brackets** to 'over-ride' BIDMAS.
  If you want to add a and b and **then** double them, you can write $2(a+b)$.
  (If you simply wrote $2a+b$, this would not be right according to BIDMAS.)

EXPRESSIONS, FORMULAE, EQUATIONS AND INEQUALITIES

These are all 'bits' of algebra, but we need to know the difference between them.

IGCSE INSIDER INFO: If you are asked for an expression and you give a formula or equation instead (or the other way round) you will **lose marks**!

- ▷ **Expression**: a fragment of algebra **with no '=' sign**.
  $3x+5$ is an expression, and so is πr^2.

- ▷ **Formula**: tells you the connection between two or more quantities; it **has an '=' sign**.
  $A = \pi r^2$ is a formula giving the area of a circle in terms of its radius.

- ▷ **Equation**: a mathematical statement **with an '=' sign**; it is only true for certain values.
  $3x+5 = 14$ is an equation which is only true when $x = 3$.

TJP TOP TIP: **EQUA**tion has **EQUA**ls in it.

- ▷ **Inequality**: a mathematical statement **with a '<', '>', '≤' or '≥' sign**.
  $3x+5 < 14$ is an inequality which is only true when $x < 3$.

TJP TOP TIP: The **wider** end of the inequality goes with the **bigger** number.
(Personally, I don't do crocodiles, but if they work for you...)

IGCSE ALGEBRA

WORDY QUESTIONS

▷ We may have to take a situation described in English and 'translate' it into algebra.

TJP TOP TIP: If in doubt, **pretend that the letters are numbers** and think about what you'd do with the numbers. **Then swap the numbers for letters**.

▶ **SKILL: convert a wordy question into algebra.**

Q: If apples cost 20p each and bananas cost 15p each, how much do a apples and b bananas cost?

A: [Suppose we had 2 apples and 3 bananas. We'd work out $20\times2+15\times3$.]
So a apples and b bananas would cost $20a+15b$ pence.

Q: Farmer Chris has g geese and h horses. How many feet do they have?

A: [If we had 5 geese (each with 2 feet) and 6 horses (each with 4 feet), they would have $2\times5+4\times6$ feet.]
So g geese and h horses have $2g+4h$ feet.

Q: If a square of side x has an area equal to its perimeter, find the value of x.

A: [If a square has side 3, its area is 3^2 and its perimeter is 3+3+3+3.]
If a square has side x, its area = x^2 and its perimeter = $x+x+x+x=4x$, so we need to solve $x^2=4x$ (which we'll come to later on).

EVALUATING EXPRESSIONS AND FORMULAE

▷ A complicated name for a simple idea with easy marks...
All you need to do is **replace all the letters with numbers and work out the answer**!

▶ **SKILL: Evaluate an expression or a formula.**
Remember to use BIDMAS and your calculator correctly (see NUMBER guide).

Q: Work out the value of $\frac{x^2}{(y+z)}$ if $x=-6, y=-3, z=12$.

A: Substitute to get $\frac{(-6)^2}{(-3+12)} = \frac{36}{9} = 4$.

NB a negative number squared **must** be positive, even if your calculator says it's not.

Also, with a division bar you need to work out the whole top line and the whole bottom line **first**, and **then** divide them.

Q: Evaluate $V = \frac{4}{3}\pi r^3$ where $r=5.73$, giving your answer correct to 2 decimal places.

A: Substitute to get $V = \frac{4}{3}\pi\times5.73^3 = 788.05$.

DEALING WITH BRACKETS (I): BASIC EXPANDING AND FACTORISING

We need to be good at taking things out of brackets and putting them back into brackets.

▶ **SKILL: Multiply out (expand) a single term by a bracket.**
Multiply each term inside the bracket by whatever is in front of the bracket.

Q: Expand $-3(4x-5)$.

A: $-3(4x-5) = -3\times4x+-3\times(-5) = -12x+15$. (NB: last term is $+15$, not -15 .)

Q: Multiply out $5a^2b(2a-3b+4ab)$.

A: $5a^2b(2a-3b+4ab) = 5a^2b\times2a+5a^2b\times(-3b)+5a^2b\times4ab$

$= 10a^3b-15a^2b^2+20a^3b^2$

▶ **SKILL: Factorise into a single bracket.**

- Find the HCF of all the terms [the biggest number and letter(s) that goes into them].
- Write down this HCF in front of the brackets.
- Divide all the original terms by this HCF and put them in the brackets.

TJP TOP TIP: You can always check your answer by expanding the brackets again... If you've done it right, the terms in the bracket should have nothing in common.

Q: Factorise $18a-27$.

A: The HCF is 9 so we get $9(2a-3)$

[Check: $9(2a-3) = 9\times2a+9\times(-3) = 18a-27$. Correct!]

Q: Factorise $8xy^2+20x^2$.

A: The HCF is $4x$ so we get $4x(2y^2+5x)$.

TJP TOP TIP: With more complicated questions, you can set out your working as for HCF and LCM (see NUMBER guide). You get the HCF by multiplying what's in the left-hand column, and the brackets contain the terms in the bottom row. See below:

Q: Factorise $70a^3b^5c^2-28ab^3c^3+42a^2b^4c^4$

A:

2	$70a^3b^5c^2$	$-28ab^3c^3$	$+42a^2b^4c^4$
7	$35a^3b^5c^2$	$-14ab^3c^3$	$+21a^2b^4c^4$
a	$5a^3b^5c^2$	$-2ab^3c^3$	$+3a^2b^4c^4$
b^3	$5a^2b^5c^2$	$-2b^3c^3$	$+3ab^4c^4$
c^2	$5a^2b^2c^2$	$-2c^3$	$+3abc^4$
	$5a^2b^2$	$-2c$	$+3abc^2$

a is the biggest factor of a^3, a, a^2
b^3 is the biggest factor of b^5, b^3, b^4
c^2 is the biggest factor of c^2, c^3, c^4

HCF = $2\times7ab^3c^2 = 14ab^3c^2$ (from the left-hand column)

Answer: $14ab^3c^2(5a^2b^2-2c+3abc^2)$ (using the bottom row)

[Check: the terms in brackets have no common factors.]

IGCSE ALGEBRA

COLLECTING LIKE TERMS

▷ In the course of a question, you may need to gather matching 'bits' together.
This is called **collecting like terms**.

TJP TOP TIP: Matching terms must have exactly the same letters and powers, but the numbers in front don't matter.

Very simply, 2 apples + 3 bananas + 5 apples + 8 bananas = 7 apples + 11 bananas.
Add up all the apples, then add up all the bananas.

▶ **SKILL: Collect like terms.**

Q: Simplify $3a+4b-5a+(-8b)$.

A: $3a+4b-5a+(-8b) = -2a-4b$.
[Go through once adding the a terms, then go through again adding the b terms.]

Q: Simplify $x^2+3x+4x+12$.

A: $x^2+3x+4x+12 = x^2+7x+12$.
[Only the x terms match; x is different from x^2, so don't group them together.]

Q: Expand and simplify $3(x+2y)-3(x-2y)$.

A: This is what I call **GROTBAG**: Get Rid Of The Brackets And Group.

$3(x+2y)-3(x-2y)$

$= 3x+6y-3x+6y$ (The last term is $+6y$ not $-6y$ because $-3\times(-2y) = 6y$.)

$= 12y$ (The x terms cancel out.)

SIMPLIFYING INDICES

▷ Another way of tidying up algebra in questions is by simplifying indices.

We use the three laws of indices:

▶	**Multiply** by **adding the indices**:	$x^4 \times x^5 = x^9$
	Divide by **subtracting the indices**:	$y^{12} \div y^2 = y^{10}$
	Do **brackets** by **multiplying the indices**:	$(z^3)^5 = z^{15}$

LEARN!

Any numbers should be worked out alongside the indices.

▶ **SKILL: Simplify algebraic indices.**

Q: Simplify $\dfrac{3x^3y^2 \times 4xy^4}{6x^2y^5}$.

A: $\dfrac{3x^3y^2 \times 4xy^4}{6x^2y^5} = \dfrac{3\times 4}{6} \times x^{3+1-2}\, y^{2+4-5} = 2x^2y$

IGCSE ALGEBRA

SOLVING LINEAR EQUATIONS

▷ A **linear equation** is one with only numbers and x terms (no higher powers or roots). '**Solve**' means find the value(s) of x that makes the equation true.

Remember to do **exactly the same thing** to each side of the equation to keep them equal.

IGCSE INSIDER INFO: If you show no working but get the right answer, you get **no marks**! So even if you can do it all in your head, write down every step.

TJP TOP TIP: To solve an equation, first of all group the x terms together if required. Then 'undo' the equation, one step at a time, to get $x = \ldots$.
The last thing to happen (by BIDMAS) is the first thing to undo.

▶ **SKILL: Solve a linear equation.**

Q: Solve $3x+7 = 19$.

A: $3x+7 = 19$ — Last thing to happen is $+7$.

$(-7) \quad (-7)$ — First, undo $+7$ with -7 .

$3x = 12$

$(\div 3) \quad (\div 3)$ — Then undo $\times 3$ with $\div 3$

$x = 4$

Q: Solve $3x+7 = x-9$.

A: $3x+7 = x-9$ — First, group the x terms together.

$(-x) \quad (-x)$ — Subtract the **smaller** of $3x$ and x , to keep things positive.

$2x+7 = -9$ — From here on it's like the previous example.

$(-7) \quad (-7)$

$2x = -16$

$(\div 2) \quad (\div 2)$

$x = -8$

If the equation contains fractions, get a common denominator before proceeding.

Q: Solve $\dfrac{2x+1}{3} = \dfrac{3x-4}{5}$.

A: $\dfrac{2x+1}{3} = \dfrac{3x-4}{5}$ — First, get a common denominator.

$\dfrac{10x+5}{15} = \dfrac{9x-12}{15}$ — Now the numerators must be equal.

$10x+5 = 9x-12$ — Group the x terms together like before.

$(-9x) \quad (-9x)$

$x+5 = -12$

$(-5) \quad (-5)$

$x = -17$

SOLVING LINEAR INEQUALITIES

▷ A **linear inequality** has a '<', '>', '≤' or '≥' sign instead of an '='.
Just treat it exactly like an equation, with one important exception:

> TJP TOP TIP: **Flip the inequality** if you **multiply** or **divide** by a **negative number**.
> This is because you are flipping the number line, so 'less than' becomes 'greater than'.

▶ **SKILL: Solve a linear inequality.**

Q: Solve $5x-4 < 11$.

A: Do the same as for an equation.

$$\begin{array}{lcl} 5x-4 & < & 11 \\ \quad(+4) & & (+4) \\ 5x & < & 15 \\ (\div 5) & & (\div 5) \\ x & < & 3 \end{array}$$

Q: Solve $-4x+7 \geq 23$.

A:

$$\begin{array}{lcl} -4x+7 & \geq & 23 \\ \quad(-7) & & (-7) \\ -4x & \geq & 16 \\ (\div -4) & & (\div -4) \\ x & \leq & -4 \end{array}$$

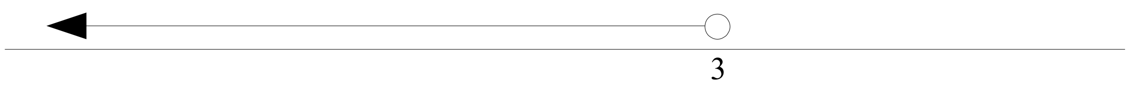
Flip the inequality because we're dividing by –4.

▶ **SKILL: Show an inequality on a number line.**

> TJP TOP TIP: Think 'more ink/less ink'
> If it's '≤' or '≥', use a solid blob ● (symbols using more ink).
> If it's '<' or '>', use a hollow blob ○ (symbols using less ink).
>
> Also, the arrowhead points the same way as the '<' or '>'.

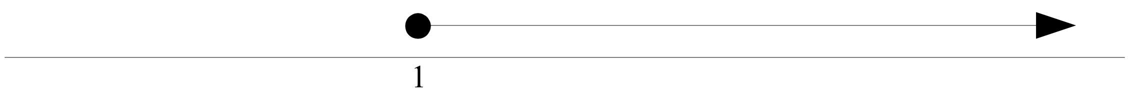
Q: Show $x < 3$ on a number line.

A:



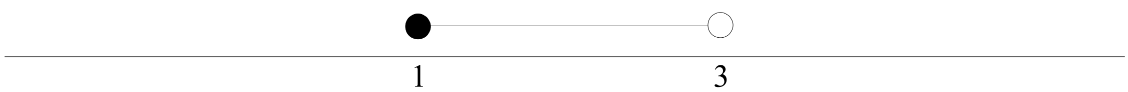
Q: Show $x \geq 1$ on a number line.

A:



Q: Show $1 \leq x < 3$ on a number line.

A:



DEALING WITH BRACKETS (II): EXPANDING QUADRATICS

▷ To **multiply a bracket by a bracket**, use **FOIL** (or **Smiley Face**).

FOIL stands for **F**irst, **O**uter, **I**nner, **L**ast.

This means:

Multiply the **F**irst terms in each bracket,	$(x+2)(x+3) \rightarrow x^2$
Multiply the **O**uter two terms in the brackets,	$(x+2)(x+3) \rightarrow 3x$
Multiply the **I**nner two terms in the brackets,	$(x+2)(x+3) \rightarrow 2x$
Multiply the **L**ast terms in each bracket.	$(x+2)(x+3) \rightarrow 6$

Then add these bits together to get the final answer: x^2+5x+6.

In other words, use **MAAD** (**M**ultiply **A**cross, **A**dd **D**own) on your **FOIL**.

If you prefer, **Smiley Face** does the same thing like this:

$(x+2)(x+3)$ [The curves tell you what bits to multiply together.]

[The eyes are just for fun...]

▶ **SKILL: Expand brackets using FOIL.**

Q: Expand $(x+3)(x-5)$.

A:
F: $x \times x = x^2$
O: $x \times (-5) = -5x$
I: $3 \times x = 3x$
L: $3 \times (-5) = -15$

Answer: $x^2+3x-5x-15 = \mathbf{x^2-2x-15}$

Q: Expand $(4x+5)(3x-2)$.

A:
F: $4x \times 3x = 12x^2$
O: $4x \times (-2) = -8x$
I: $5 \times 3x = 15x$
L: $5 \times (-2) = -10$

Answer: $12x^2-8x+15x-10 = \mathbf{12x^2+7x-10}$

Q: Work out $(x-4)^2$.

A: Warning! This is FOIL in disguise!
First, rewrite it as $(x-4)^2 = (x-4)(x-4)$.

F: $x \times x = x^2$
O: $x \times (-4) = -4x$
I: $-4 \times x = -4x$
L: $(-4) \times (-4) = 16$

Answer: $x^2-4x-4x+16 = \mathbf{x^2-8x+16}$

DEALING WITH BRACKETS (II): FACTORISING QUADRATICS

▷ To **factorise a quadratic expression into brackets**, use the 'Anti-FOIL' methods below:

▶ **SKILL: Factorise a simple quadratic into brackets (one lot of x^2).**

Q: Factorise $x^2+7x+12$.

A: Find two numbers that **add to make 7** and **multiply to make 12**.
[Start with the 'multiply' part to cut down your options. Could be 1 & 12, 2 & 6 or 3 & 4.]

The required numbers are 3 and 4.

Answer: $(x+3)(x+4)$

Q: Factorise x^2+2x-8.

A: Find two numbers that **add to make 2** and **multiply to make –8**.
[Could be 1 & 8 or 2 & 4, then decide about minus signs.]

The required numbers are –2 and 4.

[We need one +ve and one –ve number to multiply to make –8. Since they add to +2, the +ve number must 'beat' the –ve one, so it's –2 & 4, not –4 & 2.]

Answer: $(x-2)(x+4)$

IGCSE INSIDER INFO: The next type of question often comes up – don't be caught out!

Q: Factorise x^2-9x.

A: Find two numbers that **add to –9** and **multiply to 0** (you can write it as x^2-9x+0).
These numbers are 0 and –9.

Answer: $(x-0)(x-9) = x(x-9)$.

Or, note that the terms have an x in common, so the answer is immediately $x(x-9)$.

Q: Factorise x^2-9. [Note: This is called **Difference Between Two Squares**]

A: Find two numbers that **add to 0** (there are **no** lots of x) and **multiply to –9**.
These numbers are –3 and 3.

Answer: $(x-3)(x+3)$

▷ If the coefficient of x^2 is greater than 1, we use a slightly 'tweaked' method (**Fairbrother**).

▶ **SKILL: Factorise a harder quadratic into brackets (more than one lot of x^2).**

Q: Factorise $12x^2-x-6$.

A: Find two numbers that **add up to –1** and **multiply to make** 12×(–6) = **–72**.
[Could be 1 & 72, 2 & 36, 3 & 24, 4 & 18, 6 & 12 or 8 & 9.]

The required numbers are –9 and 8.

Start by writing $(12x-9)(12x+8)$ and then divide by common factors 3 and 4 to give

Answer: $(4x-3)(3x+2)$

IGCSE ALGEBRA

SOLVING QUADRATIC EQUATIONS

▷ There are **two totally different methods** for **solving quadratic equations** (which are equations with an x^2 in them, and no higher powers).

Fortunately, there is a cunning way to decide which method to use...

IGCSE INSIDER INFO:
If you are asked to **round your answer** (to 3 sig figs, 2 dp), use the **Quadratic Formula**.
If you **don't have to round your answer**, **factorise** the quadratic **into two brackets**.

Whichever you use, you **must** rearrange your equation to get **'= 0'** on the right hand side.

▶ **SKILL: Solve a quadratic equation using the formula.**

The quadratic formula is printed inside the front cover of your examination paper.

If $ax^2+bx+c = 0$ then $x = \dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$

The $\pm$ symbol is a sort of 'buy one, get one free'.
You work the formula out once with a +, then you do it all again with a –.

Yes, there are usually **two** answers to a quadratic equation...
[And equations with x^3 in them can have up to **three** answers, but they're not in IGCSE.]

TJP TOP TIP: Beware **Faulty Squares**: your calculator may claim that $-3^2 = -9$, which is wrong. A negative number squared is always **positive**.

I also recommend using the $\frac{\square}{\square}$ key on the calculator for working this out (**Big Divide**).

Be **exceptionally careful** about which numbers are positive and which are negative...

Q: Solve $4x^2-3x-5 = 0$, giving your answers to 3 significant figures.

A: Here, $a=4, b=-3, c=-5$ so $x = \dfrac{-(-3)\pm\sqrt{(-3)^2-4\times4\times(-5)}}{2\times4} = \dfrac{3\pm\sqrt{9-(-80)}}{8}$

So $x = \dfrac{3+\sqrt{89}}{8} = \mathbf{1.55}$ or $x = \dfrac{3-\sqrt{89}}{8} = \mathbf{-0.804}$.

▶ **SKILL: Solve a quadratic equation by factorising.**

Simply put the quadratic into brackets, then read off the solutions.

Q: Solve $x^2-3x-28 = 0$.

A: Find two numbers that add to –3 and multiply to –28 and hence rewrite this as $(x-7)(x+4) = 0$.

The solutions are found by considering **what number would make each bracket zero**.

So $x = \mathbf{7}$ or $x = \mathbf{-4}$.

SOLVING QUADRATIC INEQUALITIES

▷ If you are asked to solve a **quadratic inequality**, it won't usually need the quadratic formula – it'll tend to be fairly simple to work out.

BUT there's a catch! Once you have your two numbers, you have to decide if it's a **sandwich** or an **anti-sandwich**.

Here's an example to explain...

▶ **SKILL: Solve a quadratic inequality.**

Q: Solve the inequality $x^2 \le 25$.

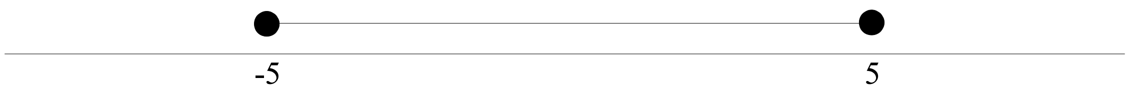
A: It's tempting to square root both sides to get $x \le 5$.

While this is true, it's not the whole answer.

For example, if $x = -10$ (which is certainly less than 5), then $x^2 = 100$ (which is not less than 25). So this value of x **doesn't work**.

In fact the answer is $-5 \le x \le 5$, which is a **sandwich** (x is sandwiched between –5 and 5).

On the number line, this looks like



Here's another example.

Q: Solve $(x-2)(x-5) > 0$.

A: Start by solving the matching **equation** $(x-2)(x-5) = 0$.
The solutions are $x = \mathbf{2}$ and $x = \mathbf{5}$.

Back to the inequality; is it true for values of x **between** 2 and 5 or **outside** 2 and 5?

Do an 'experiment' to find out. If we choose $x = 3$ (which is between 2 and 5), then $(3-2)(3-5) = -2 < 0$ which doesn't work.

So we need an **anti-sandwich**, $x < 2$ or $x > 5$.

On the number line, this looks like

TJP TOP TIP: If the inequality is 'x^2 bits **< (or ≤)** number', it's a **sandwich**. Imagine the x^2 eating the sandwich with its < (or ≤) mouth...

Otherwise it's an **anti-sandwich**.

ALGEBRAIC FRACTIONS

▷ **Algebraic fractions** are simply fractions which contain **letters** instead of/as well as numbers. We use the same standard rules as before for adding, multiplying, etc.

▶ **SKILL: Simplify algebraic fractions.**

TJP TOP TIP: Factorise everything in sight! Then cancel down any matching bits.

Q: Simplify $\frac{3x+12}{2x+8}$.

A: Factorise and then cancel down: $\frac{3x+12}{2x+8} = \frac{3(x+4)}{2(x+4)} = \frac{3}{2}$.

Q: Simplify $\frac{x^2-x-12}{x^2-2x-8}$.

A: Factorise and then cancel down: $\frac{x^2-x-12}{x^2-2x-8} = \frac{(x-4)(x+3)}{(x-4)(x+2)} = \frac{x+3}{x+2}$.

▶ **SKILL: Add or subtract algebraic fractions.**

Just get a common denominator first and then add/subtract the numerators.

Q: Work out $\frac{x-3}{2} - \frac{x+1}{5}$.

A: Common denominator is 10: $\frac{x-3}{2} - \frac{x+1}{5} = \frac{5x-15}{10} - \frac{2x+2}{10} = \frac{3x-17}{10}$.

Q: Work out $\frac{1}{x} + \frac{1}{x+1}$.

A: $\frac{1}{x} + \frac{1}{x+1} = \frac{x+1}{x(x+1)} + \frac{x}{x(x+1)} = \frac{2x+1}{x(x+1)}$.

▶ **SKILL: Solve equations involving algebraic fractions.**

Get a common denominator, then 'zap' it (multiply through by it).

Q: Solve $\frac{1}{3} + \frac{1}{x+1} = \frac{x}{3}$.

A: $\frac{x+1}{3(x+1)} + \frac{3}{3(x+1)} = \frac{x(x+1)}{3(x+1)}$ [now multiply through by $3(x+1)$]

$(x+1) + 3 = x(x+1) = x^2+x$

$4 = x^2$

$x = \pm 2$

SOLVING LINEAR SIMULTANEOUS EQUATIONS

▷ Linear simultaneous equations are two (or more) linear equations that must be true **at the same time**. 'Linear' means there are no pesky x^2 terms (or roots, etc.).

TJP TOP TIP: To solve linear simultaneous equations:

- **Match up** the number of x or y first (multiply through by a number if required).
- **SSS?** This means '**Same Sign? Subtract!**'. If the matching terms are both +ve (or both –ve) then subtract one equation from the other. Otherwise add the equations.
- **Solve** this new equation, then go back and find the **other** letter.

▶ **SKILL: Solve linear simultaneous equations.**

Q: Solve $\begin{aligned} 2x+3y &= 13 \\ 4x+11y &= 41 \end{aligned}$.

A: Match up the number of x by doubling the first equation.

$$\begin{aligned} \boxed{4x} + 6y &= 26 \\ \boxed{4x} + 11y &= 41 \end{aligned}$$

SSS? Yes! The matching x terms are both positive.

So subtract the top equation from the bottom one (to keep things positive):

$$\begin{aligned} 5y &= 15 \\ y &= 3 \end{aligned}$$

Now go back and find x; substitute for y in the easiest original equation.

$$\begin{aligned} 2x+3\times3 &= 2x+9 = 13 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

So $x = 2$ and $y = 3$. [You can check these values in the original equations.]

Q: Solve $\begin{aligned} 3x-4y &= -12 \\ 5x+6y &= -1 \end{aligned}$.

A: Match up the number of y by tripling the first equation and doubling the second one.

$$\begin{aligned} 9x \boxed{-12y} &= -36 \\ 10x \boxed{+12y} &= -2 \end{aligned}$$

SSS? No! The matching y terms have opposite signs.

So add the two equations:

$$\begin{aligned} 19x &= -38 \\ x &= -2 \end{aligned}$$

Now go back and find y; substitute for x in the easiest original equation.

$$\begin{aligned} 5\times(-2)+6y &= -10+6y = -1 \\ 6y &= 9 \\ y &= 1.5 \end{aligned}$$

So $x = -2$ and $y = 1.5$.

TJP TOP TIP: If you get **horrible answers** (not a whole number or a simple fraction), you are **probably wrong**... And you can always check your answers anyway.

SOLVING QUADRATIC SIMULTANEOUS EQUATIONS

▷ Quadratic simultaneous equations are two (or more) equations that must be true **at the same time**. 'Quadratic' means there are x^2 or y^2 terms in there.

TJP TOP TIP: To solve quadratic simultaneous equations:

- **Match up** the number of x or y first, **if possible**. [If not, read on...]
- **SSS?** This means '**Same Sign? Subtract!**'. If the matching terms are both +ve (or both –ve) then subtract one equation from the other. Otherwise add the equations.
- **Solve** this new **quadratic** equation, then go back and find the **other** letter.

▶ **SKILL: Solve simple quadratic simultaneous equations.**

Q: Solve $\begin{aligned} y &= 2x+3 \\ y &= x^2 \end{aligned}$.

A: The y terms already match! This often happens in these questions.

SSS? Yes! The matching y terms are both positive.

So subtract the top equation from the bottom one (to keep the x^2 term positive):

$y-y = x^2-(2x+3)$
$0 = x^2-2x-3$.

Now factorise this to solve:
$0 = (x-3)(x+1)$

So $x = 3$ or $x = -1$.

Now go back and find y; substitute for x in the easiest original equation, $y = x^2$.

If $x = 3$, then $y = 3^2 = 9$.
If $x = -1$, then $y = (-1)^2 = 1$.

▶ **SKILL: Solve harder quadratic simultaneous equations.**

If we can't match up x or y , we have to **substitute** instead...

Q: Solve $\begin{aligned} 2x+y &= 1 \\ x^2+y^2 &= 2 \end{aligned}$.

A: Rearrange the first equation to get $y = 1-2x$.

Now substitute this into the second equation:

$x^2+(1-2x)^2 = 2$
$x^2+1-4x+4x^2 = 2$
$5x^2-4x+1 = 2$ Put the quadratic equal to zero.
$5x^2-4x-1 = 0$ Now for a spot of Fairbrother...

Find two numbers which add to –4 and multiply to 5×(–1) = –5.
These are –5 and 1, so we get:

$(5x-5)(5x+1) = 0$ or $(x-1)(5x+1) = 0$ (cancelling down)

If $x = 1$, then $y = 1-2\times 1 = -1$.
If $x = -\frac{1}{5}$, then $y = 1-2\times -\frac{1}{5} = 1\frac{2}{5}$.

REARRANGING FORMULAE

▷ We may be asked to **rearrange a formula** to get a different letter on the left-hand side.

▶ **SKILL: Rearrange a formula where the letter appears once.**

TJP TOP TIP: Draw up a **flowchart** showing what happens to this letter, then **reverse** it. This is a really powerful and easy method! You can use it for inverse functions, too.

Q: Rearrange $A = \pi r^2$ to make r the subject.

A: Draw up a flowchart:

$r \rightarrow \boxed{^2} \rightarrow \boxed{\times\pi} \rightarrow A$ Start with r, square it, $\times\pi$, answer A.

Now reverse it:

$r \leftarrow \boxed{\sqrt{}} \leftarrow \boxed{\div\pi} \leftarrow A$ Start with A, $\div\pi$, square root it, answer r.

Read off the answer:

$r = \sqrt{(A \div \pi)}$

Q: Rearrange $t = \sqrt{\dfrac{b(m^2+a)}{e}}$ to make m the subject.

A: Flowchart:

$m \rightarrow \boxed{^2} \rightarrow \boxed{+a} \rightarrow \boxed{\times b} \rightarrow \boxed{\div e} \rightarrow \boxed{\sqrt{}} \rightarrow t$

Reverse it:

$m \leftarrow \boxed{\sqrt{}} \leftarrow \boxed{-a} \leftarrow \boxed{\div b} \leftarrow \boxed{\times e} \leftarrow \boxed{^2} \leftarrow t$

Read off:

$m = \sqrt{\dfrac{e t^2}{b} - a}$.

Make sure that BIDMAS matches your flowchart so things get done in the right order.

▶ **SKILL: Rearrange a formula where the letter appears twice.**

If the letter appears twice, first group them together in one place!
Remember: **Expand, Group, Factorise**.

Q: Rearrange $\dfrac{y+x}{y-x} = C$ to make y the subject.

A: $y+x = C(y-x)$ Multiply both sides by the denominator

$y+x = Cy-Cx$ **Expand** the brackets

$x+Cx = Cy-y$ **Group** the y terms on one side

$x(1+C) = y(C-1)$ **Factorise** (put back into brackets again)

$\dfrac{x(C+1)}{(C-1)} = y$ Divide to finish

IGCSE ALGEBRA

FUNCTIONS

▷ A **function** is a mathematical 'black box' that does something to whatever you put into it.

There are **two ways** of writing a function, but they are **both the same**:
$f(x) = $ (expression) or $f : x \rightarrow$ (expression)

▶ **SKILL: Evaluate a function.**

Q: If $f(x) = 2x^2+3$, find $f(4),\ f(-1),\ f(a)$ in turn.

A: $f(4) = 2\times4^2+3 = 2\times16+3 = 35$ Simply replace the x with 4.

$f(-1) = 2\times(-1)^2+3 = 2\times1+3 = 5$ (Remember BIDMAS!)

$f(a) = 2a^2+3$ Replace the x with a; nothing else to do.

▶ **SKILL: Find the domain and range of a function.**

TJP TOP TIP: Some terminology:
Doma**in** = the allowed numbers that go **in** to a function.
Range = the numbers that come out of a function.

Numbers **not allowed** in the **domain** almost always involve:
- Square roots of negative numbers
- Division by zero

The **range** is the 'height' of the graph; the y-values it can take.

Q: Find the domain of $f(x) = \sqrt{x+5}$.

A: We can't square root a negative number (try it on your calculator!) so we need
$x+5 \geq 0$

Answer: Domain is $x \geq -5$.

Q: Which values of x must be excluded from the domain of $g : x \rightarrow \frac{1}{x^2-25}$?

A: We can't divide by zero so we must exclude any values where
$x^2-25 = 0$.

Answer: Exclude $x = 5,\ x = -5$ from the domain.

Q: Find the range of $h(x) = (x-2)^2-7$.

A: It's tempting to multiply this out, but don't do it!
Consider how this graph has been translated: right 2 and down 7.
It thus has a minimum at (2, –7), so the height of the graph is always –7 or above.

Answer: Range is $h(x) \geq -7$.

Q: What is the range of $f : x \rightarrow 4x+1$ when the domain is $0 \leq x \leq 10$?

A: Here, the domain is restricted to x values from 0 to 10.
This is a straight-line graph, so its range (the possible y values) go from
$4\times0+1 = 1$ up to $4\times10+1 = 41$.

Answer: Range is $1 \leq f \leq 41$.

▷ The **inverse** of a function 'undoes' the original function; it's a flowchart put into reverse. The inverse of $f(x)$ is written as $f^{-1}(x)$.

▶ **SKILL: Find the inverse of a function.**
If x occurs only once, do a flow chart and reverse it.

Q: If $f(x) = 2(x^3-17)$, work out $f^{-1}(x)$.

A: Flowchart: $x \rightarrow \boxed{^3} \rightarrow \boxed{-17} \rightarrow \boxed{\times 2} \rightarrow f(x)$

Reverse it: $f^{-1}(x) \leftarrow \boxed{\sqrt[3]{}} \leftarrow \boxed{+17} \leftarrow \boxed{\div 2} \leftarrow x$

Answer: $f^{-1}(x) = \sqrt[3]{\left(\frac{x}{2}+17\right)}$.

If x occurs **more than once**, do the following steps:
- Write the function as $y = ...$
- Change all x into y and y into x
- Rearrange to make y the subject; the right-hand side is the inverse function.

Q: Find the inverse function of $g(x) = \frac{x}{x-2}$.

A: $y = \frac{x}{x-2}$ becomes $x = \frac{y}{y-2}$.
$x(y-2) = y$
$xy-2x = y$
$xy-y = 2x$
$y(x-1) = 2x$
$y = \frac{2x}{x-1}$ so $g^{-1}(x) = \frac{2x}{x-1}$.

If $f(x) = f^{-1}(x)$, then the function is **self inverse**; it 'undoes' itself. E.g. $f(x) = 6-x$.

▷ A **composite function** is what you get when you put one function 'inside' another function. **WARNING**: This is **not** the same as multiplying the functions!

TJP TOP TIP: Always write in the invisible brackets: $fg(x)$ really means $f(g(x))$. Then work from the inside outwards.

▶ **SKILL: Work out composite functions.**

Q: If $f(x) = x^2$ and $g(x) = x+2$, find $fg(x)$ and $gf(x)$.

A: $fg(x) = f(g(x))$ — Put in the invisible brackets!
$= f(x+2)$ — Start from the inside, replacing $g(x)$ with $x+2$.
$= (x+2)^2 = x^2+4x+4$ — Then put $x+2$ through the 'squaring' function, $f(x)$.

$gf(x) = g(f(x))$ — Put in the invisible brackets!
$= g(x^2)$ — Start from the inside, replacing $f(x)$ with x^2.
$= x^2+2$ — Then put x^2 through the '+2' machine, $g(x)$.

Sneaky trick question: $f\,f^{-1}(x) = x$ because $f^{-1}(x)$ does the opposite of $f(x)$.

PROPORTION

▷ Two quantities are **proportional** if they change so that one of them is always a **fixed multiple** of the other. This means that if you double one quantity, the other one is doubled, too.

There are several words and symbols meaning exactly the same thing:

A **varies as** B; A is (**directly**) **proportional** to B; A ∝ B; A **= k**B

There are five more possibilities listed in the syllabus, namely:

A is proportional to (or varies as) the square of B	$A \propto B^2$
A is proportional to (or varies as) the cube of B	$A \propto B^3$
A is inversely proportional to (or varies inversely as) B	$A \propto 1/B$
A is inversely proportional to (or varies inversely as) the square of B	$A \propto 1/B^2$
A is proportional to (or varies as) the square root of B	$A \propto \sqrt{B}$

▶ **SKILL: Solve proportion problems.**

Good news: proportion questions are remarkably predictable...

- Write down the proportion relation and swap the '∝' for '= k×'
- Use the given data to get the value of k and write down the master formula.
- Use this formula forwards.
- Use this formula in reverse.

Q: If p varies as the square of q, and $p = 20$ when $q = 2$, find

(a) p in terms of q (b) p when $q = 10$ (c) q when $p = 605$

A: (a) Here, $p \propto q^2$ is rewritten as $p = kq^2$.

Then substitute $p = 20,\ q = 2$ to get $20 = k \times 2^2$, so $k = 5$.

Master formula is: $p = 5q^2$.

(b) When $q = 10,\ p = 5 \times 10^2 = 5 \times 100 = 500$.

(c) When $p = 605,\ 605 = 5q^2$ so $121 = q^2$. Therefore $q = \pm 11$.

Q: If y is inversely proportional to x and $y = 4$ when $x = 3$, find

(a) y in terms of x (b) y when $x = 2$ (c) x when $y = 3$

A: (a) We rewrite $y \propto \frac{1}{x}$ as $y = k \times \frac{1}{x}$.

Then substitute $x = 3,\ y = 4$ to get $4 = k \times \frac{1}{3}$, so $k = 12$.

Master formula is $y = 12 \times \frac{1}{x}$ (or $y = \frac{12}{x}$ if you prefer).

(b) When $x = 2,\ y = 12 \times \frac{1}{2} = 6$.

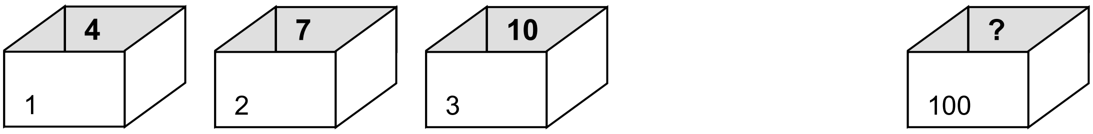
(c) When $y = 3,\ 3 = 12 \times \frac{1}{x}$ so $\frac{3}{12} = \frac{1}{4} = \frac{1}{x}$. Therefore $x = 4$.

IGCSE ALGEBRA

SEQUENCES

▷ A **sequence** is a **list of numbers**, usually following some pattern.

Imagine a row of boxes, numbered 1, 2, 3, etc., and each box contains a number in the sequence:

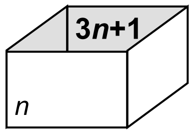


Mathematically we say that $t_1 = 4$, $t_2 = 7$, $t_3 = 10$, etc.

What is in box 100? We could just draw all the boxes and count all the way up to 100…

Or we could find a formula for what is in box *n*, where *n* stands for *n*-ything.

In this case, $t_n = 3n+1$



t_n means '**what's in the *n*th box**' or '**the *n*th term**' (the *n*th number in the list). Sometimes you will see u_n or another letter used, but it's exactly the same idea.

Now to find what's in box no. 100, we simply put *n* = 100 into the formula.

So $t_{100} = 3n+1 = 3\times100+1 = 301$.

▶ **SKILL: Find a formula for a linear sequence.**

TJP TOP TIP: Find two things from your sequence:

- the **gap** between next-door numbers (–ve if they go down, +ve if they go up)
- the **0th term;** the number that would come **before** the first number in the list.

Then **t_n = Gap×*n* + 0th term** **LEARN!!!** [**only** works if the gap is fixed]

Q: Find a formula for the sequence 7, 5, 3, 1, -1, …

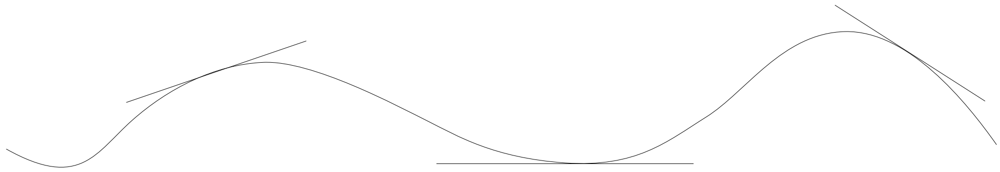
A: Gap = –2 [numbers go down by 2]
0th term = 9 [would come before the 7]
So $t_n = -2n+9$.

▷ Some **well-known sequences** – learn to recognise them!

1, 4, 9, 16, 25, …	Square numbers [$t_n = n^2$]
1, 3, 6, 10, 15, …	Triangle numbers [$t_n = \frac{1}{2}n(n+1)$]
1, 8, 27, 64, 125, …	Cube numbers [$t_n = n^3$]
2, 3, 5, 7, 11, 13, …	Prime numbers [no pattern to the gaps]
1, 1, 2, 3, 5, 8, 13, …	Fibonacci numbers [add the previous two to get the next one]

DIFFERENTIATION

▷ **Differentiation** is all about finding the **gradient** (slope) **of a curve**. The gradient will change as we move along the curve, so the gradient is a formula involving x .



TJP TOP TIP: Lots of different words and symbols, all meaning the same thing:

Find the **gradient**; **differentiate**; find the **derivative**; find **dy/dx**; find **f'(x)**

Find the gradient formula by doing this to each term: $x^n \Rightarrow n x^{n-1}$

In English; multiply by the power, then decrease the power by 1.
[Also, x terms go to numbers, numbers disappear.]

▶ **SKILL: Differentiate an expression or a function.**

Q: Differentiate $4x^5 - 6x^2 + 9x - 13$.

A: $5 \times 4x^{5-1} - 2 \times 6x^{2-1} + 9 = 20x^4 - 12x + 9$

If we have '1 over' terms, rewrite them using **negative indices**, then differentiate.

Q: If $y = \frac{1}{x^3} - \frac{4}{x}$, find $\frac{dy}{dx}$.

A: Rewrite $y = x^{-3} - 4x^{-1}$, so

$$\frac{dy}{dx} = -3 \times x^{-3-1} - (-1) \times 4x^{-1-1} = -3x^{-4} + 4x^{-2} = \frac{-3}{x^4} + \frac{4}{x^2} .$$

▷ Where a graph reaches a **maximum** (top of a hill) or a **minimum** (bottom of a valley), the **gradient must be zero** ('cos we're not going uphill or downhill – see the graph above).

Maximum and minimum points are called **turning points**, and we can find them by working out the **derivative**, setting it equal to **zero** and solving this equation.

To decide whether we have a maximum or a minimum point:
(a) use the graph's shape; x^2 graphs have a minimum, $-x^2$ graphs have a maximum,
(b) with **two** turning points (x^3 graphs), the **higher** one is the **maximum**.

▶ **SKILL: Find the turning points of a graph.**

Q: Find and identify any turning points of the graph $y = x^3 - 6x^2 + 10$.

A: $\frac{dy}{dx} = 3x^2 - 12x = 3x(x-4) = 0$ at any turning points.

At $x = 0$, $y = 0^3 - 6 \times 0^2 + 10 = 10$
At $x = 4$, $y = 4^3 - 6 \times 4^2 + 10 = -22$

(0, 10) is a maximum and (4, –22) is a minimum (it's lower).

▶ **SKILL: Find where the gradient takes a certain value.**

Q: Find the co-ordinates of the point where the graph $y = -x^2+5x+8$ has a gradient of –3.

A: Get the gradient and set it equal to –3.

$$\frac{dy}{dx} = -2x+5 = -3$$
$$-2x = -8$$
$$x = 4$$

When $x = 4,\ y = -4^2+5\times4+8 = -16+20+8 = 12$.

So the point is (4, 12).

▷ **Kinematics** is the study of the **motion** of objects – it comes from the same origin as 'cinema' (moving pictures) and 'kinetic energy' (motion energy).

> **TJP TOP TIP**: **Kinematics** questions need the following facts:
>
> Velocity = derivative of position. $v = \frac{dx}{dt}$
>
> Acceleration = derivative of velocity. $a = \frac{dv}{dt}$

▶ **SKILL: Solve a kinematics question.**

Q: If a ball is thrown in the air so that its height is given by $h = 5t-5t^2$ metres, find (i) its velocity and (ii) its acceleration after t seconds.

Hence find its maximum height.

A: The position is given by h, so we differentiate this once to get v and again to get a.

(i) $v = \frac{dh}{dt} = 5-10t$ m/s (ii) $a = \frac{dv}{dt} = -10$ m/s².

To find its maximum height, note that its velocity $v = 5-10t$ is zero when $t = 0.5$. [It reaches the top when its velocity drops to zero.]

At this time, the height $h = 5\times\frac{1}{2} - 5\times\left(\frac{1}{2}\right)^2 = 1.25$ metres.

IGCSE ALGEBRA

CONTENTS