GCE A level Further Mathematics (9FM0) – Shadow Paper (Set 1)
9FM0-01 AL Core Pure 1

October 2021 Shadow Paper mark scheme

Please note that this mark scheme is not the one used by examiners for making scripts. It is intended more as a guide, indicating where marks are given for correct answers. As such, it may not show follow-through marks (marks that are awarded despite errors being made) or special cases.

It should also be noted that for many questions, there may be alternative methods of finding correct solutions that are not shown here – they will be covered in the formal mark scheme from the original paper.

This document is intended for guidance only and may differ significantly from the examiners' final mark scheme for the original paper which was published in December 2021.

Guidance on the use of codes within this document
M1 – method mark. This mark is generally given for an appropriate method in the context of the question. This mark is given for showing your working and may be awarded even if working is incorrect. A1 – accuracy mark. This mark is generally given for a correct answer following correct working. B1 – working mark. This mark is usually given when working and the answer cannot easily be separated. Some questions require all working to be shown; in such questions, no marks will be given for an answer with no working (even if it is a correct answer).

Question	Scheme	Marks	AOs
1(a) **Way 1**	$\det \mathbf{T} = \left(-5\sqrt{2}\right)\times\left(-5\sqrt{2}\right)-\left(-5\sqrt{2}\right)\times 5\sqrt{2} = \ldots \Rightarrow c = \sqrt{\det \mathbf{T}} = \ldots$	M1	3.1a
	$c = 10$	A1	1.1b
	$\Rightarrow \mathbf{T}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \Rightarrow \cos\theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \ldots$	M1	1.1b
	$\left(\cos\theta < 0, \sin\theta < 0 \Rightarrow \text{ Quadrant 3 so}\right) \quad \theta = 225°$	A1	1.1b
		(4)	
Way 2	$\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = c\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -5\sqrt{2} & 5\sqrt{2} \\ -5\sqrt{2} & -5\sqrt{2} \end{pmatrix}$	M1	3.1a
	Achieves both the equations $c\cos\theta = -5\sqrt{2}$ and $c\sin\theta = -5\sqrt{2}$	A1	1.1b
	$\frac{c\sin\theta}{c\cos\theta} = \frac{-5\sqrt{2}}{-5\sqrt{2}} \Rightarrow \tan\theta = 1 \Rightarrow \theta = \ldots$	M1	1.1b
	$\theta = 225°$ and $c = 10$	A1	1.1b
		(4)	
(b)	Area of A' = area of $A \times c^2$ so area of A is $75 \div 100$ The area of the triangle $A = 3k^2$ so $3k^2 = \frac{75}{100}$	M1	1.1b
	$k = \frac{1}{2}$	A1ft	2.2a
		(2)	
			(6 marks)

Question	Scheme	Marks	AOs
2 (a)	$\sin\left(\frac{x}{2}\right)\arctan\left(\frac{x}{2}\right) \approx \left(\frac{x}{2}-\frac{\left(\frac{x}{2}\right)^3}{3!}+\frac{\left(\frac{x}{2}\right)^5}{5!}-\ldots\right)\left(\frac{x}{2}-\frac{\left(\frac{x}{2}\right)^3}{3}+\frac{\left(\frac{x}{2}\right)^5}{5}-\ldots\right)=\ldots$ or $\left(\frac{x}{2}-\frac{x^3}{48}+\frac{x^5}{3840}-\ldots\right)\left(\frac{x}{2}-\frac{x^3}{24}+\frac{x^5}{160}-\ldots\right)=\ldots$	M1	2.2a
	$=\frac{x^2}{4}-\frac{x^4}{32}+\frac{19x^6}{4608}$	A1	1.1b
	$-1\le\frac{x}{2}\le1\Rightarrow-2\le x\le2$	B1	3.2b
		(3)	
(b)	$\int\frac{8}{x^3}\left(\frac{x^2}{4}-\frac{x^4}{32}+\frac{19x^6}{4608}\right)=\int\frac{2}{x}-\frac{x}{4}+\frac{19x^3}{576}=A\ln x+Bx^2+Cx^4$ where A, B and $C\neq0$	M1	3.1a
	$2\ln x-\frac{x^2}{8}+\frac{19x^4}{2304}$	A1ft	1.1b
	$\left(\ln3-\frac{3}{8}+\frac{19}{256}\right)-\left(0-\frac{1}{8}+\frac{19}{2304}\right) = \ln3-\frac{53}{288}$	A1	2.2a
		(3)	
(c)	Calculator = awrt 0.89755	B1	1.1b
		(1)	
			(7 marks)

Question	Scheme	Marks	AOs
3	$y = 3x + 2 \Rightarrow x = \dfrac{y-2}{3}$	B1	3.1a
	$a\left(\dfrac{y-2}{3}\right)^3 + a\left(\dfrac{y-2}{3}\right)^2 + b\left(\dfrac{y-2}{3}\right) + c\ (=0)$ or $(3x+2)^3 - k(3x+2)^2 + 6(3x+2) + 28\ (=0)$	M1	3.1a
	$ay^3 - 3ay^2 + 9by + (27c - 18b + 4a) = 0$ or $27x^3 + (54-9k)x^2 + (54-12k)x + (48-4k) = 0$	M1	1.1b
(i)	Divides by a for $y^3 - 3y^2 + \dfrac{9b}{a}y + \left(\dfrac{27c-18b+4a}{a}\right) = 0$ and states $k = 3$ from coefficient of y^2 or $27x^3 + (54-9k)x^2 + (54-12k)x + (48-4k) = 0 \Rightarrow 27 = 54 - 9k$ and solves to $k = 3$	A1	1.1b
(ii)	Equates coefficient of y and constant $\dfrac{9b}{a} = 6$ and $\dfrac{27c-18b+4a}{a} = 28\left(\Rightarrow \dfrac{27c-18b}{a} = 24\right)$ and solves simultaneously to find b or c in terms of a or Substitutes $k = 3$ into $27x^3 + (54-9k)x^2 + (54-12k)x + (48-4k) = 0$	M1	3.1a
	(any positive integer) multiple of $a = 3\ \ b = 2\ \ c = 4$ (Note: $27x^3 + (54-9k)x^2 + (54-12k)x + (48-4k) = 0$ is $27x^3 + 27x^2 + 18x + 36 = 0$)	A1	1.1b
		(6)	
			(6 marks)

(i)	**Alternative** At least two of $\alpha+\beta+\gamma=-\frac{a}{a}=(-1)$ $\quad \alpha\beta+\alpha\gamma+\beta\gamma=\frac{b}{a}$ $\quad \alpha\beta\gamma=-\frac{c}{a}$	B1	3.1a
	New sum $=3(\alpha+\beta+\gamma)+6=-\left(\frac{-k}{1}\right)\Rightarrow 3(-1)+6=k\Rightarrow k=3$	A1	3.1a
(ii)	New pair sum $=9(\alpha\beta+\alpha\gamma+\beta\gamma)+12(\alpha+\beta+\gamma)+12=\frac{6}{1}$ $\Rightarrow 9\left(\frac{b}{a}\right)-12+12=6\Rightarrow 9\left(\frac{b}{a}\right)-12+12=6\Rightarrow 9\left(\frac{b}{a}\right)=6\Rightarrow b=...$	M1	1.1b
	New product $27(\alpha\beta\gamma)+18(\alpha\beta+\alpha\gamma+\beta\gamma)+12(\alpha+\beta+\gamma)+8=\frac{-28}{1}$ $\Rightarrow 27\left(-\frac{c}{a}\right)+18\left(\frac{2}{3}\right)+12(-1)+8=-28\Rightarrow c=...$	M1	1.1b
	$b=\frac{2}{3}a \quad c=\frac{4}{3}a$	A1	1.1b
	(any positive integer) multiple of $a=3$ $b=2$ $c=4$	A1	1.1b
		(6)	
			(6 marks)

Question	Scheme	Marks	AOs
4(i)	$\mathbf{R}^{-1}=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{pmatrix}$	B1	2.4
		(1)	
(ii) (a)	$\lambda=10$	B1	2.2a
	$s=3,\ t=2$	B1	2.2a
(b)	Inverse matrix $=\frac{1}{10}\begin{pmatrix}8 & 7 & -12\\ 2 & -2 & 2\\ -4 & -1 & 6\end{pmatrix}$	B1 ft	3.1a
		(3)	
(iii)	A complete method to find the determinant of the matrix and set equal to zero.	M1	1.1b
	Determinant = $1(0)-1(0)+1(\tan\theta\times1-\tan2\theta\times-1)=0$	A1	1.1b
	Uses compound angle formula to achieve $\tan\theta+\frac{2\tan\theta}{1-\tan^2\theta}=0\Rightarrow\tan^3\theta-3\tan\theta=0$ leading to $\theta=...$	M1	3.1a
	$\theta=-\frac{\pi}{3},0,\frac{\pi}{3}$	A1	1.1b
		(4)	
			(8 marks)

Question	Scheme	Marks	AOs
5(i)	$\int 3\mathrm{e}^{(4-2x)}\,\mathrm{d}x = -\frac{3}{2}\mathrm{e}^{(4-2x)}$	B1	1.1b
	$\int_2^{\infty} 3\mathrm{e}^{(4-2x)}\,\mathrm{d}x = \lim_{t\to\infty}\left[\left(-\frac{3}{2}\mathrm{e}^{(4-2t)}\right)-\left(-\frac{3}{2}\mathrm{e}^{(4-2\times 2)}\right)\right]$	M1	2.1
	$= \frac{3}{2}$	A1	1.1b
		(3)	
(ii)(a)	When $t = 0$, $h = 380 - 2\times 0 + \frac{0^2}{20} - 15\sin\left(\frac{\pi}{10}\times 0\right) = 380$ When $t = 40$, $h = 380 - 2\times 40 + \frac{40^2}{20} - 15\sin\left(\frac{\pi}{10}\times 40\right)$ $= 380 - 80 + 80 - 15\sin(4\pi) = 380$	B1	1.2
		(1)	
(ii)(b)	Mean height $= \frac{1}{40}\int_0^{40}\left(380 - 2t + \frac{t^2}{20} - 15\sin\left(\frac{\pi}{10}t\right)\right)\mathrm{d}t$	B1	1.2
	$= \frac{1}{40}\left[\left(380t - t^2 + \frac{t^3}{60} + \frac{150}{\pi}\cos\left(\frac{\pi}{10}t\right)\right)\right]_0^{40} = \frac{1}{40}[...]$	M1	1.1b
	$= \frac{1}{40}\left[\left(380\times 40 - 40^2 + \frac{40^3}{60} + \frac{150}{\pi}\cos(4\pi)\right) - \left(\frac{150}{\pi}\cos(0)\right)\right] = \frac{1100}{3}$ * cso	A1*	2.1
		(3)	
			(7 marks)

Question	Scheme	Marks	AOs
6 (a)	Solves $25m^2 - 20m + 4 = 0 \Rightarrow m = ...$	M1	3.1b
	$m = 0.4$ (repeated root)	A1	1.1b
	$p = e^{0.4t}\left(At + B\right)$	A1ft	1.1b
	$t = 0,\ p = 1500 \Rightarrow A = ...\ (= 1500)$	M1	3.4
	$t = 1,\ p = 2300 \Rightarrow 2300e^{0.4\times 1}\left(1500 + B\times 1\right)$	M1	1.1b
	$B = 40$ (accept or $B = 42$, or $B = 41.7$)	dM1	3.4
	$p = e^{0.4t}\left(1500 + 40t\right)$ (accept $p = e^{0.4t}\left(1500 + 42t\right)$, $p = e^{0.4t}\left(1500 + 41.7t\right)$ …)	A1	1.1b
		(7)	
(b)	Population $p = e^{0.4\times 3}\left(1500 + 40\times 3\right)$ (accept $p = e^{0.4\times 3}\left(1500 + 42\times 3\right)$, $p = e^{0.4\times 3}\left(1500 + 41.7\times 3\right)$ …) **sc** from misread 2021 to 2023; $p = e^{0.4\times 2}\left(1500 + 40\times 2\right)$ etc can award M1 but not the subsequent A1	M1	3.4
	Population = awrt 5380 (for $B = 40$), 5400 (for $B = 42$ or $B = 41.7$)	A1	2.2b
		(2)	
(c)	For example As the number of rats becomes very high, the island will not be able to sustain the population.	B1	3.5b
		(1)	
			(10 marks)

Question	Scheme	Marks	AOs
7(a)	$\begin{pmatrix}3\\2\\-2\end{pmatrix}.\begin{pmatrix}2\\-2\\1\end{pmatrix}=6-4-2=0$ and $\begin{pmatrix}1\\1\\0\end{pmatrix}.\begin{pmatrix}2\\-2\\1\end{pmatrix}=2-2+0=0$	M1	1.1b
	As $2\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is perpendicular to both direction vectors (two non-parallel vectors) of Π then it must be perpendicular to Π	A1	2.2a
		(2)	
(b)	$\begin{pmatrix}x\\y\\z\end{pmatrix}.\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}7\\3\\2\end{pmatrix}.\begin{pmatrix}2\\-2\\1\end{pmatrix}\Rightarrow \ldots$	M1	1.1a
	$2x-2y+z=10$	A1	2.2a
		(2)	
(c)	$\dfrac{\lvert 2(-4+5\lambda)-2(-6)+1(6-2\lambda)-10\rvert}{\sqrt{2^2+(-2)^2+1^2}}=8\Rightarrow \lambda=\ldots$	M1	3.1a
	$\lambda=-3$ and $\lambda=3$	A1	1.1b
	$\mathbf{r}=\begin{pmatrix}-4\\-6\\6\end{pmatrix}-3\begin{pmatrix}5\\0\\-2\end{pmatrix}=\ldots$ or $\mathbf{r}=\begin{pmatrix}-4\\-6\\6\end{pmatrix}+3\begin{pmatrix}5\\0\\-2\end{pmatrix}=\ldots$	M1	1.1b
	$(-19,-6,12)$ and $(11,-6,0)$	A1	2.2a
		(4)	
(d)	Points $(-19,-6,12)$ and $(11,-6,0)$ are equidistant from Π but on opposite sides of it. Their midpoint lies on Π. Or Values of λ found for points in **(c)** are of equal magnitude but opposite sign, so $\lambda=0$ gives point of intersection of l and Π.	M1	
	$(-4,-6,6)$	A1	
		(2)	
			(10 marks)

Question	Scheme		Marks	AOs
8(a)	Volume of mixture = (8 + 2) litres therefore Rate of antifreeze out $= 0.4 \times \frac{x}{10}$ litres per second		M1	3.3
	$\frac{\mathrm{d}x}{\mathrm{d}t} = 0.3 - \frac{x}{25}$		A1	1.1b
			(2)	
(b)	Rearranges $\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{x}{25} = 0.3$ and attempts integrating factor IF = $\mathrm{e}^{\int \frac{1}{25}\mathrm{d}t} = \ldots$	Separates the variables $\int \frac{1}{7.5 - x}\,\mathrm{d}x = \int \frac{1}{25}\,\mathrm{d}t$ $\Rightarrow \ldots$	M1	3.1a
	$x\mathrm{e}^{\frac{t}{25}} = \int 0.3\mathrm{e}^{\frac{t}{25}}\,\mathrm{d}t \Rightarrow x\mathrm{e}^{\frac{t}{25}} = \lambda \mathrm{e}^{\frac{t}{25}}(+c)$	Integrates to the form $\lambda \ln(7.5 - x) = \frac{1}{25}t(+c)$	M1	1.1b
	$x\mathrm{e}^{\frac{t}{25}} = 7.5\mathrm{e}^{\frac{t}{25}} + c$	$-\ln(7.5 - x) = \frac{1}{25}t + c$	A1ft	1.1b
	$t = 0,\ x = 2 \Rightarrow c = \ldots$		M1	3.4
	$x = \frac{7.5\mathrm{e}^{\frac{t}{25}} - 5.5}{\mathrm{e}^{\frac{t}{25}}} = 4$ rearranges to achieve $\mathrm{e}^{\frac{t}{5}} = \alpha$ and solves to find a value for t or $x = 7.5 - 5.5\mathrm{e}^{-\frac{t}{25}} = 4$ rearranges to achieve $\mathrm{e}^{-\frac{t}{25}} = \beta$ and solves to find a value for t	$-\ln(7.5 - 4) = \frac{1}{25}t - \ln 5.5$ Leading to a value for t	M1	3.4
	$t =$ awrt 11.3 seconds		A1	2.2b
			(6)	
(c)	$x = \frac{7.5\mathrm{e}^{\frac{t}{25}} - 5.5}{\mathrm{e}^{\frac{t}{25}}} = \frac{7.5 - 5.5\mathrm{e}^{-\frac{t}{25}}}{1}$ so $= \lim_{t \to \infty} x = \frac{7.5 - 5.5 \times 0}{1} = 7.5$ Then 7.5 as a percentage of 10…		M11	3.5a
	$c = 75$ (%)		A1	2.2b
			(2)	
			(10 marks)	

Question	Scheme	Marks	AOs
9(a)	$\int \frac{x^2}{\sqrt{x^2+4}}\,\mathrm{d}x \to \int \mathrm{f}(u)\,\mathrm{d}u$ Uses the substitution $x = 2\sinh u$ fully to achieve an integral in terms of u only, including replacing the dx	M1	3.1a
	$\int \frac{4\sinh^2 u}{\sqrt{4\sinh^2 u+4}} \times 2\cosh u\,(\mathrm{d}u) = \int \frac{4\sinh^2 u}{\sqrt{\sinh^2 u+1}}\cosh u\,(\mathrm{d}u)$	A1	1.1b
	Uses correct identities $\cosh^2 u - 1 = \sinh^2 u$ and $\cosh 2u = 1 + 2\sinh^2 u$ to achieve an integral of the form $A\int(\cosh 2u \mp 1)\,\mathrm{d}u$	M1	3.1a
	Integrates to achieve $A\left(\frac{1}{2}\sinh 2u \mp u\right)(+c)$	M1	1.1b
	Uses the identity $\sinh 2u = 2\sinh u\cosh u$ and $\cosh^2 u - 1 = \sinh^2 u$ and $x = 2\sinh u$ to reach $\sinh 2u = 2\times\frac{x}{2}\sqrt{\frac{x^2}{4}+1} = \frac{x}{2}\sqrt{x^2+4}$	M1	2.1
	$\frac{x}{2}\sqrt{x^2+4} - 2\operatorname{arsinh}\left(\frac{x}{2}\right) + k$ * cso	A1*	1.1b
		(6)	
(b)	Uses integration by parts the correct way around to achieve $\int x\operatorname{arsinh}\left(\frac{x}{2}\right)\mathrm{d}x = Px^2\operatorname{arsinh}\left(\frac{x}{2}\right) - Q\int \frac{x^2}{\sqrt{x^2+4}}\,\mathrm{d}x$	M1	2.1
	$= \frac{1}{2}x^2\operatorname{ar\,sin}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{x^2}{\sqrt{x^2+4}}\,\mathrm{d}x$	A1	1.1b
	$= \frac{1}{2}x^2\operatorname{arsinh}\left(\frac{x}{2}\right) - \frac{1}{2}\left(\underline{\frac{x}{2}\sqrt{x^2+4} - 2\operatorname{arsinh}\left(\frac{x}{2}\right)}\right)$	B1ft	2.2a
	Uses the limits x =0 and x = 4 the correct way around and subtracts $8\operatorname{arsinh}\left(\frac{4}{2}\right) - \frac{1}{2}\left(\underline{\frac{4}{2}\sqrt{4^2+4} - 2\operatorname{arsinh}\left(\frac{4}{2}\right)}\right) - 0$	dM1	1.1b
	$8\operatorname{arsinh} 2 - \frac{1}{2}\left(\underline{4\sqrt{5} - 2\operatorname{arsinh}2}\right) - 0$ $= 9\ln\left(2+\sqrt{5}\right) - 2\sqrt{5}$ *	A1*	1.1b
		(5)	
		(11 marks)	