

GCE A level Further Mathematics (9FM0) – Shadow Paper (Set 1)
9FM0-02 AL Core Pure 2

October 2021 Shadow Paper mark scheme

Please note that this mark scheme is not the one used by examiners for making scripts. It is intended more as a guide, indicating where marks are given for correct answers. As such, it may not show follow-through marks (marks that are awarded despite errors being made) or special cases.

It should also be noted that for many questions, there may be alternative methods of finding correct solutions that are not shown here – they will be covered in the formal mark scheme from the original paper.

This document is intended for guidance only and may differ significantly from the examiners' final mark scheme for the original paper which was published in December 2021.

Guidance on the use of codes within this document
M1 – method mark. This mark is generally given for an appropriate method in the context of the question. This mark is given for showing your working and may be awarded even if working is incorrect. A1 – accuracy mark. This mark is generally given for a correct answer following correct working. B1 – working mark. This mark is usually given when working and the answer cannot easily be separated. Some questions require all working to be shown; in such questions, no marks will be given for an answer with no working (even if it is a correct answer).

Question	Scheme	Marks	AOs
1(a) (i) (ii)	$\left \frac{z_1}{z_2} \right = \frac{\sqrt{3}}{2}$	B1	1.1b
	$\arg\left(\frac{z_1}{z_2}\right) = \frac{5\pi}{12} - \frac{\pi}{4} = \frac{\pi}{6}$ o.e.	B1	1.1b
		(2)	
(b) (i) (ii)	$n = 6$	B1ft	2.2a
	$ w^n = \left(\text{'their'} \left \frac{z_1}{z_2} \right \right)^{\text{their } n}$	M1	1.1b
	$ w^n = \frac{27}{64}$	A1	1.1b
		(3)	
(5 marks)			

Question	Scheme		Marks	AOs
2	$\begin{pmatrix} 2 & -3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} x \\ mx+c \end{pmatrix} = \begin{pmatrix} X \\ mX+c \end{pmatrix}$ leading to an equation in x, m, c and X		M1	3.1a
	$2x - 3(mx + c) = X$ and $4(mx + c) = mX + c$		A1	1.1b
	$4(mx + c) = m(2x - 3(mx + c)) + c$ leading to equations		M1	2.1
	$4m = 2m - 3m^2$	$4c = -3mc + c$		
	$m = 0$ or $m = \frac{2}{3}$	$c = 0$ or $m = -1$ if $c \neq 0$	dM1	1.1b
	Only combination of m and c (where $m \neq 0$) that satisfies both equations is $m = \frac{2}{3}$ and $c = 0$ If the line does not pass through the origin (ie $c \neq 0$) then the only value of m that satisfies both equations is $m = 0$ (ie gradient is zero).		A1	2.4
			(5)	
(5 marks)				

Question	Scheme	Marks	AOs
3(a)	$f'(x) = A(1-x^2)^{-\frac{1}{2}} \quad f''(x) = Bx(1-x^2)^{-\frac{3}{2}} \text{ and}$ $f'''(x) = C(1-x^2)^{-\frac{3}{2}} + Dx^2(1-x^2)^{-\frac{5}{2}} \text{ or } \frac{C(1-x^2)^{\frac{3}{2}} + Dx^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$	M1	2.1
	$f'(x) = -(1-x^2)^{-\frac{1}{2}} \text{ or } \frac{-1}{\sqrt{1-x^2}} \quad f''(x) = -x(1-x^2)^{-\frac{3}{2}} \text{ or } \frac{-x}{(1-x^2)^{\frac{3}{2}}}$ $f'''(x) = -(1-x^2)^{-\frac{3}{2}} - 3x^2(1-x^2)^{-\frac{5}{2}} \text{ or } \frac{-1}{(1-x^2)^{\frac{3}{2}}} - \frac{3x^2}{(1-x^2)^{\frac{5}{2}}}$ and from quotient rule $\frac{-(1-x^2)^{\frac{3}{2}} - 3x^2(1-x^2)^{\frac{1}{2}}}{(1-x^2)^3}$	A1	1.1b
	Finds $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and applies the formula $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0)$ $\left\{ f(0) = \frac{\pi}{2}, f'(0) = -1, f''(0) = 0, f'''(0) = -1 \right\}$	M1	1.1b
	$f(x) = \frac{\pi}{2} - x - \frac{x^3}{6}$	A1	1.1b
		(4)	
(b)	$\arccos\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{2} - \frac{\sqrt{2}}{2} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{6} = \frac{\pi}{4} \Rightarrow \pi = \dots$	M1	1.1b
	$\pi = \frac{13\sqrt{2}}{6} \text{ o.e.}$	A1ft	2.2b
		(2)	
(6 marks)			

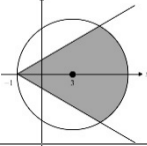
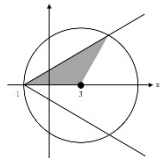
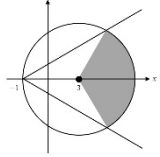
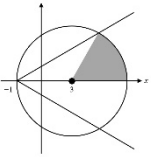
Question	Scheme	Marks	AOs
4(a)	A complete attempt to find $\sum (4r-1)^3$ by expanding and using three of the standard summation formulae.	M1	3.1a
	$\sum_{r=1}^n (4r-1)^3 = \sum_{r=1}^n (64r^3 - 48r^2 + 12r - 1)$ $= 64 \sum_{r=1}^n r^3 - 48 \sum_{r=1}^n r^2 + 12 \sum_{r=1}^n r - \sum_{r=1}^n 1$	M1	1.1b
	$= 64 \times \frac{1}{4} n^2 (n+1)^2 - 48 \times \frac{1}{6} n(n+1)(2n+1) + 12 \times \frac{1}{2} n(n+1) - n$ $= 16n^2 (n+1)^2 - 8n(n+1)(2n+1) + 6n(n+1) - n \text{ oe}$	M1 A1	1.1b 1.1b
	$16n^4 + 16n^3 - 2n^2 - 3n$	A1	2.1
		(5)	
(b)	Solves quartic equation or inequality $16n^4 + 16n^3 - 2n^2 - 3n - 1500000 = 0$ or $16n^4 + 16n^3 - 2n^2 - 3n - 1500000 > 0$ and obtains value 17.255...	M1	1.1b
	$n = 18$ (or $n \geq 18$; condone $n > 17$ but not $n \geq 17$)	A1	
		(2)	
(c)	Attempts to expand either $(4n+3)^3 - (4n-1)^3$ or $16(n+2)^4 + 16(n+2)^3 - 2(n+2)^2 - 3(n+2) - (16n^4 + 16n^3 - 2n^2 - 3n)$ or $16(n+1)^4 + 16(n+1)^3 - 2(n+1)^2 - 3(n+1) - (16(n+1)^4 + 16(n+1)^3 - 2(n+1)^2 - 3(n+1))$ oe	M1	
	Reaches $192n^2 + 96n + 28 = 28828$ oe and obtains $n = 12$	A1	
	103 823 and 132 651 (accept 47^3 and 51^3)	A1	
		(3)	
(10 marks)			

Question	Scheme	Marks	AOs
5(a)	$\frac{dy}{dx} = \frac{\lambda}{1+\beta x^2}$ where $\lambda > 0$ and $\beta > 0$ and $\beta \neq 1$ Alternatively $3 \tan y = x \Rightarrow \frac{dx}{dy} = \alpha \sec^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\alpha \sec^2 y}$	M1	1.1b
	Concludes that $\frac{dy}{dx} \neq 0$ therefore C has no stationary points; or tries to solve $\frac{dy}{dx} = 0$ and ends up with a contradiction e.g. $-1 = 0$ therefore C has no stationary points.	A1	2.4
		(2)	
(b)	$\frac{dy}{dx} = \frac{3}{9+(\sqrt{3})^2} = \dots \left\{ \frac{1}{4} \right\}$	M1	1.1b
	Tangent gradient = $\frac{dy}{dx}$ and $y - \frac{\pi}{6} = \frac{dy}{dx} \times (x - \sqrt{3})$ Alternatively $\frac{\pi}{6} = \frac{dy}{dx}(\sqrt{3}) + c \Rightarrow c = \dots \left\{ \frac{\pi}{6} - \frac{1}{4} \right\}$ and then $y = mx + c$	M1	1.1b
	Normal gradient = $-\frac{1}{m}$ and $y - \frac{\pi}{6} = m_n(x - \sqrt{3})$ or $\frac{\pi}{6} = m_n(\sqrt{3}) + c \Rightarrow c = \dots \left\{ \frac{\pi}{6} + 4\sqrt{3} \right\}$ and then $y = m_n x + c$	M1	1.1b
	For tangent: $y = 0 \Rightarrow$ $0 - \frac{\pi}{6} = \frac{1}{4}(x_Q - \sqrt{3}) \Rightarrow x_Q = \dots \left\{ \sqrt{3} - \frac{4\pi}{6} \text{ or } \sqrt{3} - \frac{2\pi}{3} \right\}$ and, for normal: $y = 0 \Rightarrow 0 - \frac{\pi}{6} = -4(x_R - \sqrt{3}) \Rightarrow x_R = \dots \left\{ \sqrt{3} + \frac{\pi}{24} \right\}$	dM1	3.1a
	Area = $\frac{1}{2} \times (x_R - x_Q) \times y_P = \frac{1}{2} \left(\left(\sqrt{3} + \frac{\pi}{24} \right) - \left(\sqrt{3} - \frac{2\pi}{3} \right) \right) \times \frac{\pi}{6}$	M1	1.1b
	Area $\frac{17}{288} \pi^2$	A1	2.1
		(6)	
(8 marks)			

Question	Scheme	Marks	AOs
6(a)	$y = r \sin \theta = 5 \cos 3\theta \sin \theta$ Leading to $\frac{dy}{d\theta} = \alpha \cos \theta \cos 3\theta + \beta \sin \theta \sin 3\theta$	M1	3.1a
	$\frac{dy}{d\theta} = 5 \cos \theta \cos 3\theta - 15 \sin \theta \sin 3\theta$	A1	1.1b
	Attempt to solve $5 \cos \theta \cos 3\theta - 15 \sin \theta \sin 3\theta = 0$ Attempts derivation/use of $\cos \theta = 4\cos^3 \theta - 3\cos \theta$ and/or $\sin 3\theta = 3\sin \theta - 4\sin^3 \theta$ $5 \cos \theta (4\cos^3 \theta - 3\cos \theta) - 15 \sin \theta (3\sin \theta - 4\sin^3 \theta) = 0$ Attempts to reach equation in $\sin \theta$ or $\cos \theta$ only	M1	3.1a
	$16 \cos^4 \theta - 18 \cos^2 \theta - 3 = 0$ or $16 \sin^4 \theta - 14 \sin^2 \theta + 1 = 0$ oe	A1	1.1b
	Substitutes $\theta \{ \pm 0.28391 \dots \}$ (ignore other angles not in range $-\frac{\pi}{6} \leq \theta \leq \frac{\pi}{6}$) into $y = r \sin \theta = 5 \cos 3\theta \sin \theta$ $h = 2 \times \{0.92252 \dots\}$	M1	3.1a
	$= 1.85$ (to 3 sf)	A1	1.1b
		(6)	
(b)	$A = \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} [5 \cos 3\theta]^2 d\theta = \frac{25}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \cos^2 3\theta d\theta$ or maybe $25 \int_0^{\frac{\pi}{6}} \cos^2 3\theta d\theta$	M1	3.4
	$\cos^2 3\theta = \frac{1}{2} + \frac{1}{2} \cos 6\theta \Rightarrow$ $A = \frac{25}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2} \cos 6\theta \right) d\theta$ or $A = 25 \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} + \frac{1}{2} \cos 6\theta \right) d\theta$ $\int \left(\frac{1}{2} + \frac{1}{2} \cos 6\theta \right) d\theta = \alpha \theta \pm \beta \sin 6\theta$	M1	3.1a
	$= \frac{25}{4} \left[\theta + \frac{1}{6} \sin 6\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$ oe or $= \frac{25}{2} \left[\theta + \frac{1}{6} \sin 6\theta \right]_0^{\frac{\pi}{6}}$ oe	A1	1.1b
	Using limits $\theta = -\frac{\pi}{6}$ and $\theta = \frac{\pi}{6}$ or $\theta = 0$ and $\theta = \frac{\pi}{6}$ as appropriate and subtracts correct way round provided integration is attempted $= \frac{25}{4} \left(\frac{\pi}{6} + \frac{1}{6} \sin \pi \right) - \frac{25}{4} \left(-\frac{\pi}{6} + \frac{1}{6} \sin(-\pi) \right)$ or	M1	1.1b

	$= \frac{25}{2} \left(\frac{\pi}{6} + \frac{1}{6} \sin \pi \right) - \frac{25}{2} \left(\frac{0}{6} + \frac{1}{6} \sin 0 \right)$		
	Mass = $30 \times 19 \times \left\{ \text{area} = \frac{25\pi}{12} \right\}$	M1	3.4
	3730 g or $\frac{2375\pi}{2}$	A1	3.2a
		(6)	
(c)	$a = 5$ (ie unchanged from C)	B1	3.5b
	b an integer > 3 , $k = \frac{\pi}{2b}$ (so, eg, $b = 4$ and $k = \frac{\pi}{8}$, etc.)	B1	3.5b
		(2)	
(14 marks)			

Question	Scheme		Marks	AOs
7(a)	x co-ordinate of P is $\frac{1}{2}\left(\mathrm{e}^{\frac{1}{2}\ln 10}+\mathrm{e}^{-\frac{1}{2}\ln 10}\right)$	$\ln\left(x+\sqrt{x^2-1}\right)=\frac{1}{2}\ln 10$	B1	1.2
	$x=\frac{1}{2}\left(\sqrt{10}+\frac{1}{\sqrt{10}}\right)\Rightarrow x=...$	$x+\sqrt{x^2-1}=\sqrt{10}$ $\sqrt{x^2-1}=\sqrt{10}-x$ $x^2-1=10-2\sqrt{10}x+x^2\Rightarrow x=...$	M1	1.1b
	$x=\frac{11\sqrt{10}}{20}$ or $\frac{11}{2\sqrt{10}}$ *		A1 *	2.2a
			(3)	
(b)	Volume = $\pi\int_0^{\frac{1}{2}\ln 10}\cosh^2 y\,\mathrm{d}y$		B1	2.5
	$\{\pi\}\int\left(\frac{\mathrm{e}^y+\mathrm{e}^{-y}}{2}\right)^2\mathrm{d}y=\{\pi\}\int\left(\frac{\mathrm{e}^{2y}+2+\mathrm{e}^{-2y}}{4}\right)\mathrm{d}y$ or $\{\pi\}\int\frac{1}{2}\sinh 2y+\frac{1}{2}\mathrm{d}y$		M1	3.1a
	$\frac{1}{4}\left(\frac{1}{2}\mathrm{e}^{2y}+2y-\frac{1}{2}\mathrm{e}^{-2y}\right)$ or $\frac{1}{4}\cosh 2y+\frac{1}{2}y$		dM1 A1	1.1b 1.1b
	Use limits $y=0$ and $y=\frac{1}{2}\ln 10$ and subtracts the correct way round $\frac{\pi}{4}\left(\frac{1}{2}\mathrm{e}^{\ln 10}+\ln 10-\frac{1}{2}\mathrm{e}^{-\ln 10}\right)=\frac{\pi}{4}\left(5+\ln 10-\frac{1}{20}\right)...$		M1	1.1b
	$=\frac{\pi}{80}(99+\ln 10)$ *		A1 *	1.1b
			(6)	
(9 marks)				

Question	Scheme	Marks	AOs
8(i)	$ z = \sqrt{(\sqrt{6} + \sqrt{2})^2 + (\sqrt{6} - \sqrt{2})^2} = \dots \{\sqrt{16} \text{ or } 4\} \text{ and}$ $\arg(z) = \tan^{-1} \left(\frac{\sqrt{6} - \sqrt{2}}{\sqrt{6} + \sqrt{2}} \right) = \dots \left\{ \frac{\pi}{12} \right\}$ Can be implied by $r = 4e^{\frac{\pi}{12}i}$	M1 A1	3.1a 1.1b
	Adding multiples of $\frac{2\pi}{3}$ to their argument $z = 4e^{\frac{\pi}{12}i} \times e^{\frac{2\pi k}{3}i} \text{ or } z = 4 \left[\cos \left(\frac{\pi}{12} + \frac{2\pi k}{3} \right) + i \sin \left(\frac{\pi}{12} + \frac{2\pi k}{3} \right) \right]$	M1	1.1b
	$z = re^{\left(\theta + \frac{2\pi}{3}\right)i}, re^{\left(\theta + \frac{4\pi}{3}\right)i} \text{ o.e. or } z = re^{\left(\theta - \frac{2\pi}{3}\right)i}, re^{\left(\theta - \frac{4\pi}{3}\right)i} \text{ o.e.}$	A1ft	1.1b
	$z = 4e^{\frac{9\pi}{12}i}, 4e^{\frac{17\pi}{12}i} \text{ o.e. or } z = 4e^{-\frac{7\pi}{12}i}, 4e^{-\frac{15\pi}{12}i} \text{ o.e.}$	A1	1.1b
		(5)	
(ii)(a)	Circle centre (0, 3), radius 4, or pair of lines with point on (0, -1) and angles above and below x axis equal. 	B1	1.1b
	Fully correct 	B1	1.1b
		(2)	
(ii)(b)	Finds area of triangle $\text{Area} = \frac{1}{2} \times 4^2 \times \sin \left(\frac{2\pi}{3} \right) \{ = 4\sqrt{3} \}$ Or finds area of sector $\text{Area} = \frac{1}{2} \times 4^2 \times \frac{2\pi}{3} \text{ or } \frac{1}{2} \times 4^2 \times \frac{\pi}{3}$  or similar  	M1	1.1b
	Adds appropriate areas eg $2 \left(\frac{1}{2} \times 4^2 \times \sin \left(\frac{2\pi}{3} \right) \right) + 2 \left(\frac{1}{2} \times 4^2 \times \frac{\pi}{3} \right) \text{ oe}$	M1	3.1a
	Reaches $\frac{8}{3}(3\sqrt{3} + 2\pi) *$	A1 *	1.1b
		(3)	

(10 marks)

Question	Scheme	Marks	AOs
9(a)	$\frac{1}{1-z}$	B1	2.2a
		(1)	
(b)(i)	$1+z+z^2+z^3+\dots$ $=1+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^2+\left(\frac{1}{2}(\cos\theta+i\sin\theta)\right)^3+\dots$ $=1+\frac{1}{2}(\cos\theta+i\sin\theta)+\frac{1}{4}(\cos 2\theta+i\sin 2\theta)+\frac{1}{8}(\cos 3\theta+i\sin 3\theta)+\dots$	M1	3.1a
	$\frac{1}{1-z}=\frac{1}{1-\frac{1}{2}(\cos\theta+i\sin\theta)}\times\frac{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}{1-\frac{1}{2}\cos\theta+\frac{1}{2}i\sin\theta}$ or $\frac{1}{1-z}=\frac{2}{2-(\cos\theta+i\sin\theta)}\times\frac{2-(\cos\theta-i\sin\theta)}{2-(\cos\theta-i\sin\theta)}$	M1	3.1a
	$1+\frac{1}{2}(\cos\theta)+\frac{1}{4}\cos(\sin 2\theta)+\frac{1}{8}(\cos 3\theta)+\dots=\frac{1-\frac{1}{2}\cos\theta}{\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2}$ or $1+\frac{1}{2}(\cos\theta)+\frac{1}{4}(\cos 2\theta)+\frac{1}{8}(\cos 3\theta)+\dots=\frac{2-\cos\theta}{(2-\cos\theta)^2+(\sin\theta)^2}$	M1	2.1
	$\left(1-\frac{1}{2}\cos\theta\right)^2+\left(\frac{1}{2}\sin\theta\right)^2=1-\cos\theta+\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta$ $=\frac{5}{4}-\cos\theta$ or $(2-\cos\theta)^2+(\sin\theta)^2=4-4\cos\theta+\cos^2\theta+\sin^2\theta$ $=5-4\cos\theta$	M1	1.1b
	$\frac{1}{2}\cos\theta+\frac{1}{4}\cos 2\theta+\frac{1}{8}\cos 3\theta+\dots=\frac{1-\frac{1}{2}\cos\theta}{\frac{5}{4}-\cos\theta}-1$ $=\frac{4-2\cos\theta}{5-4\cos\theta}-\frac{5-4\cos\theta}{5-4\cos\theta}=\frac{2\cos\theta-1}{5-4\cos\theta}$	A1*	1.1b

	Alternative $1 + z + z^2 + z^3 + \dots$ $= 1 + \left(\frac{1}{2}(\cos \theta + i \sin \theta)\right) + \left(\frac{1}{2}(\cos \theta + i \sin \theta)\right)^2 + \left(\frac{1}{2}(\cos \theta + i \sin \theta)\right)^3 + \dots$ $= 1 + \frac{1}{2}(\cos \theta + i \sin \theta) + \frac{1}{4}(\cos 2\theta + i \sin 2\theta) + \frac{1}{8}(\cos 3\theta + i \sin 3\theta) + \dots$	M1	3.1a
	$\frac{1}{1-z} = \frac{1}{1-\frac{1}{2}e^{i\theta}} \times \frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{2}e^{-i\theta}}$	M1	3.1a
	$\frac{1-\frac{1}{2}e^{-i\theta}}{1-\frac{1}{4}e^{i\theta}-\frac{1}{4}e^{-i\theta}+\frac{1}{4}} = \frac{4-2e^{-i\theta}}{5-2(e^{i\theta}+e^{-i\theta})} = \frac{4-2(\cos \theta - i \sin \theta)}{5-2(2 \cos \theta)}$	M1	2.1
	Select the real part $\frac{4-2 \cos \theta}{5-4 \cos \theta}$ and subtract 1	M1	1.1b
	$\frac{1}{2} \cos \theta + \frac{1}{4} \cos 2\theta + \frac{1}{8} \cos 3\theta + \dots = \frac{4-2 \cos \theta}{5-4 \cos \theta} - \frac{5-4 \cos \theta}{5-4 \cos \theta} = \frac{2 \cos \theta - 1}{5-4 \cos \theta}$	A1*	1.1b
		(5)	
(b)(ii)	$\frac{2 \cos \theta - 1}{5-4 \cos \theta} = -\frac{2}{7} \Rightarrow \cos \theta = \left\{-\frac{1}{2}\right\} \text{ and } \operatorname{re}(z) = \frac{1}{2} \cos \theta$	M1	3.1a
	$-\frac{1}{4}$	A1	1.1b
		(2)	
(8 marks)			