Section 2: Further differentiation

Solutions to Exercise

1. (i) $y = x^3 + 6x^2 + 9x$

$\frac{dy}{dx} = 3x^2 + 12x + 9$

(ii) $\frac{dy}{dx} = 0$

$3x^2 + 12x + 9 = 0$

$x^2 + 4x + 3 = 0$

$(x+1)(x+3) = 0$

$x = -1$ or $x = -3$

When x = -1, $y = (-1)^3 + 6(-1)^2 + 9 \times -1 = -1 + 6 - 9 = -4$

When x = -3, $y = (-3)^3 + 6(-3)^2 + 9 \times -3 = -27 + 54 - 27 = 0$

The stationary points are (-1, -4) and (-3, 0)

(iii)

x	x < -3	x = -3	-3 < x < -1	x = -1	x > -1
$\frac{dy}{dx}$	+ve /	0 —	-ve \	0 —	+ve /

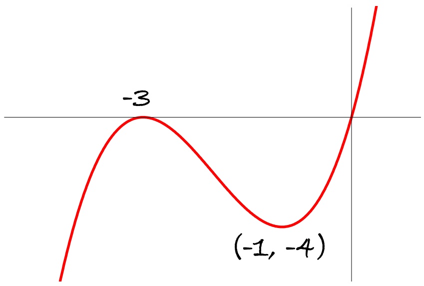
The point (-3, 0) is a maximum point.
The point (-1, -4) is a minimum point.

(iv) $y = x^3 + 6x^2 + 9x$

$= x(x^2 + 6x + 9)$

$= x(x+3)^2$

The graph cuts the x-axis at x = 0 and x = -3 (repeated).
The graph cuts the y-axis at y = 0.



2. (i) $y = 2x + x^2 - 4x^3$

$\frac{dy}{dx} = 2 + 2x - 12x^2$

At stationary points, $\frac{dy}{dx} = 0$

$2 + 2x - 12x^2 = 0$

$1 + x - 6x^2 = 0$

$6x^2 - x - 1 = 0$

$(3x+1)(2x-1) = 0$

$x = -\frac{1}{3}$ or $x = \frac{1}{2}$

When $x = -\frac{1}{3}$, $y = 2\left(-\frac{1}{3}\right) + \left(-\frac{1}{3}\right)^2 - 4\left(-\frac{1}{3}\right)^3 = -\frac{2}{3} + \frac{1}{9} + \frac{4}{27} = \frac{-18+3+4}{27} = -\frac{11}{27}$

When $x = \frac{1}{2}$, $y = 2\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 - 4\left(\frac{1}{2}\right)^3 = 1 + \frac{1}{4} - \frac{1}{2} = \frac{3}{4}$

The stationary points are $\left(-\frac{1}{3}, -\frac{11}{27}\right)$ and $\left(\frac{1}{2}, \frac{3}{4}\right)$.

x	$x < -\frac{1}{3}$	$x = -\frac{1}{3}$	$-\frac{1}{3} < x < \frac{1}{2}$	$x = \frac{1}{2}$	$x > \frac{1}{2}$
$\frac{dy}{dx}$	-ve ╲	0 ―	+ve ╱	0 ―	-ve ╲

$\left(-\frac{1}{3}, -\frac{11}{27}\right)$ is a minimum point.

$\left(\frac{1}{2}, \frac{3}{4}\right)$ is a maximum point.

(ii) $y = 2x + x^2 - 4x^3$

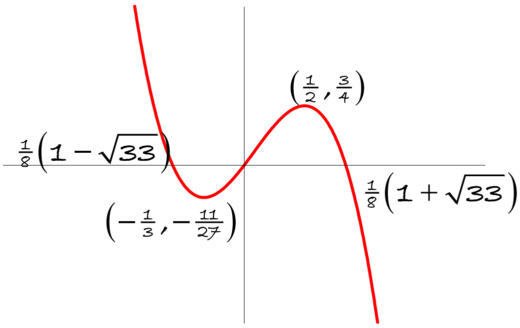
$= x(2 + x - 4x^2)$

$= -x(4x^2 - x - 2)$

The curve cuts the x-axis at $x = 0$ and at the points satisfying $4x^2 - x - 2 = 0$.

For this quadratic equation, $a = 4, b = -1, c = -2$

Using the quadratic formula, $x = \frac{1 \pm \sqrt{1 - 4 \times 4 \times -2}}{8} = \frac{1 \pm \sqrt{33}}{8}$



3. $y = x^3 - 3x^2 + 6$

$\frac{dy}{dx} = 3x^2 - 6x$

At stationary points, $\frac{dy}{dx} = 0$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

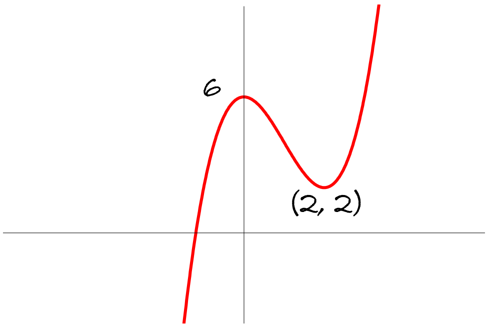
When $x = 0$, $y = 6$

When $x = 2$, $y = 2^3 - 3 \times 2^2 + 6 = 8 - 12 + 6 = 2$

The stationary points are (0, 6) and (2, 2).

x	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$\frac{dy}{dx}$	+ve /	0 —	-ve \	0 —	+ve /

(0, 6) is a maximum point.
(2, 2) is a minimum point.



4. (i) $y = (x+1)(x-3)^3$

$= (x+1)(x^3 - 9x^2 + 27x - 27)$

$= x^4 - 9x^3 + 27x^2 - 27x + x^3 - 9x^2 + 27x - 27$

$= x^4 - 8x^3 + 18x^2 - 27$

(ii) $\frac{dy}{dx} = 4x^3 - 24x^2 + 36x$

At stationary points, $4x^3 - 24x^2 + 36x = 0$

$x^3 - 6x^2 + 9x = 0$

$x(x^2 - 6x + 9) = 0$

$x(x-3)^2 = 0$

$x = 0$ or $x = 3$

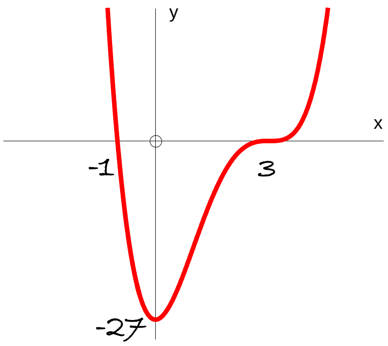
When $x = 0$, $y = (0+1)(0-3)^3 = 1 \times -27 = -27$
When $x = 3$, $y = 0$
The stationary points are (0, -27) and (3, 0).

(iii)

x	$x < 0$	$x = 0$	$0 < x < 3$	$x = 3$	$x > 3$
$\frac{dy}{dx}$	-ve ╲	0 ―	+ve ╱	0 ―	+ve ╱

(0, -27) is a minimum.
(3, 0) is a point of inflection.

(iv) When $y = 0$, $x = -1$ or $x = 3$



5. $y = x^4 - 2x^3$

$\frac{dy}{dx} = 4x^3 - 6x^2$

At stationary points, $4x^3 - 6x^2 = 0$

$x^2(2x-3) = 0$

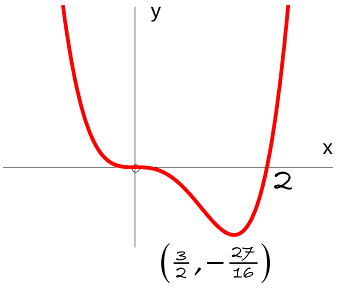
$x = 0$ or $x = \frac{3}{2}$

When $x = 0$, $y = 0$

When $x = \frac{3}{2}$, $y = \left(\frac{3}{2}\right)^4 - 2\left(\frac{3}{2}\right)^3 = \frac{81}{16} - \frac{27}{4} = -\frac{27}{16}$

x	$x < 0$	$x = 0$	$0 < x < \frac{3}{2}$	$x = \frac{3}{2}$	$x > \frac{3}{2}$
$\frac{dy}{dx}$	-ve ╲	0 ―	-ve ╲	0 ―	+ve ╱

So (0, 0) is a point of inflection, and $\left(\frac{3}{2}, -\frac{27}{16}\right)$ is a minimum point.



6. $y = x^3 + px^2 + q$

$\frac{dy}{dx} = 3x^2 + 2px$

At stationary points, $\frac{dy}{dx} = 0$

$3x^2 + 2px = 0$

$x(3x + 2p) = 0$

$x = 0$ or $x = -\frac{2p}{3}$

Since there is a minimum point at x = 4, $-\frac{2p}{3} = 4 \quad \Rightarrow p = -6$

The curve is therefore $y = x^3 - 6x^2 + q$.

The point (4, -11) lies on the curve, so $-11 = 4^3 - 6 \times 4^2 + q$

$\left(\frac{1}{2}, \frac{3}{4}\right)$ $-11 = 64 - 96 + q$

$q = 21$

The equation of the curve is $y = x^3 - 6x^2 + 21$.

The other stationary point is at x = 0, so the maximum point is (0, 21).

7. (i) $y = x^3 + ax^2 + bx + c$

The graph passes through the point (1, 1)

so $1 = 1 + a + b + c$

$a + b + c = 0$

(ii) $\frac{dy}{dx} = 3x^2 + 2ax + b$

Stationary points are when $3x^2 + 2ax + b = 0$

There is a stationary point when x = -1, so $3(-1)^2 + 2a \times -1 + b = 0$

$3 - 2a + b = 0$

$2a - b = 3$

There is a stationary point when x = 3, so $3\times 3^2 + 2a\times 3 + b = 0$

$$27 + 6a + b = 0$$

$$6a + b = -27$$

(iii) $a + b + c = 0$ (1)

$2a - b = 3$ (2)

$6a + b = -27$ (3)

Adding (2) and (3): $8a = -24 \quad \Rightarrow a = -3$

Substituting into (2) gives: $b = 2a - 3 = -6 - 3 = -9$

Substituting into (1) gives: $c = -a - b = 9 + 3 = 12$

$a = -3, b = -9, c = 12$

8. $\dfrac{dy}{dx} = 3x^2 - 10x + 3$

At a stationary point $\dfrac{dy}{dx} = 0$

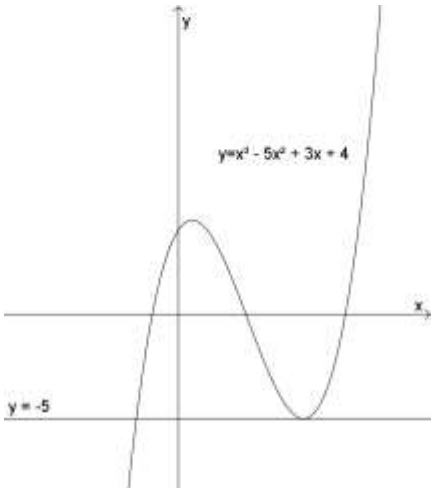
$3x^2 - 10x + 3 = 0$

$(3x - 1)(x - 3) = 0$

x = 1/3 or 3.

The minimum point is when x = 3 (from the sketch)

At this point $y = x^3 - 5x^2 + 3x + 4 = 27 - 45 + 9 + 4 = -5$



The line y = - 5 crosses the curve again. To the left of this crossing point, y takes values below those at the minimum stationary point.

$$x^3 - 5x^2 + 3x + 4 = -5$$

$$x^3 - 5x^2 + 3x + 9 = 0$$

x = 3 is a root so (x – 3) is a factor.

$x^3 - 5x^2 + 3x + 9 = 0$

$(x-3)(x^2 - 2x - 3) = 0$

$(x-3)(x-3)(x+1) = 0$

y takes values below that at the minimum stationary point for $x < -1$.

9. $\frac{dy}{dx} = 60x^2 - 114x + 54$

At a stationary point $\frac{dy}{dx} = 0$

$60x^2 - 114x + 54 = 0$

$10x^2 - 19x + 9 = 0$

$(10x - 9)(x-1) = 0$

The equation has two roots so there are two stationary points.

10. (i)

$y = 10 - 3x - x^2$

$\frac{dy}{dx} = -3 - 2x$

$\frac{d^2y}{dx^2} = -2$

(ii)

$y = 3x(x^2 - 2x) = 3x^3 - 6x^2$

$\frac{dy}{dx} = 9x^2 - 12x$

$\frac{d^2y}{dx^2} = 18x - 12$

(iii)

$y = (2x+5)(x^2 - 3x) = 2x^3 - x^2 - 15x$

$\frac{dy}{dx} = 6x^2 - 2x - 15$

$\frac{d^2y}{dx^2} = 12x - 2$

11. (i)

$$y = -2x^3 + 6x^2 + x - 7$$
$$\frac{dy}{dx} = -6x^2 + 12x + 1$$
$$\frac{d^2y}{dx^2} = -12x + 12$$

(ii)

at $x=1$, $\frac{dy}{dx} = -6(1)^2 + 12(1) + 1 = 7$

at $x=3$, $\frac{dy}{dx} = -6(3)^2 + 12(3) + 1 = -17$

at $x=-1$, $\frac{dy}{dx} = -6(-1)^2 + 12(-1) + 1 = -17$

(iii)

at $x=1$, $\frac{d^2y}{dx^2} = -12(1) + 12 = 0$

at $x=3$, $\frac{d^2y}{dx^2} = -12(3) + 12 = -24$

at $x=-1$, $\frac{d^2y}{dx^2} = -12(-1) + 12 = 24$