

Section 6: Sequences and proof

Solutions to Exercise

1. If n is odd, n^3 and n^2 are both odd, so $n^3 - n^2$ is even.
If n is even, n^3 and n^2 are both even, so $n^3 - n^2$ is even.
Therefore for all positive integers n , $n^3 - n^2$ is always even.

2. For three consecutive integers, at least one will be even, and one will be a multiple of 3.
Therefore the product of the three integers is both a multiple of 2 and a multiple of 3,
and so is a multiple of 6.

3. Any odd number can be written as $2n + 1$, where n is an integer.
So the square of an odd number can be written as $(2n + 1)^2 = 4n^2 + 4n + 1$

$$= 4n(n + 1) + 1$$
One of n and $n + 1$ is even, so $n(n + 1)$ is a multiple of 2 and therefore $4n(n + 1)$ is a multiple of 8.
So $4n(n + 1) + 1$ is one more than a multiple of 8. So the square of any odd number is always 1 more than a multiple of 8.

4. (i) n th term $= 3n - 1$
 1^{st} term $= 3 \times 1 - 1 = 2$
 2^{nd} term $= 3 \times 2 - 1 = 5$
 3^{rd} term $= 3 \times 3 - 1 = 8$
 4^{th} term $= 3 \times 4 - 1 = 11$
Sequence is 2, 5, 8, 11,

- (ii) n th term $= n^2 - 1$
 1^{st} term $= 1^2 - 1 = 0$
 2^{nd} term $= 2^2 - 1 = 4 - 1 = 3$
 3^{rd} term $= 3^2 - 1 = 9 - 1 = 8$
 4^{th} term $= 4^2 - 1 = 16 - 1 = 15$
Sequence is 0, 3, 8, 15,

- (iii) n th term $= 3n^2 - 2n + 1$
 1^{st} term $= 3 \times 1^2 - 2 \times 1 + 1 = 3 - 2 + 1 = 2$
 2^{nd} term $= 3 \times 2^2 - 2 \times 2 + 1 = 12 - 4 + 1 = 9$
 3^{rd} term $= 3 \times 3^2 - 2 \times 3 + 1 = 27 - 6 + 1 = 22$
 4^{th} term $= 3 \times 4^2 - 2 \times 4 + 1 = 48 - 8 + 1 = 41$
Sequence is 2, 9, 22, 41,

5. (i) Each term increases by 3, so the general term must involve $3n$.
 n th term $= 3n - 1$.

(ii) Each term decreases by 2, so the general term must involve $-2n$.
 n th term $= 12 - 2n$

6. (i) The sequence has n th term $an^2 + bn + c$.

Terms	3	9	17	27	39
Differences		6	8	10	12
Second differences			2	2	2

So $a = 1$

Terms	3	9	17	27	39
an^2	1	4	9	16	25
$bn + c$	2	5	8	11	14

The values of $bn + c$ go up by 3 each time, so $b = 3$, and $c = -1$

The n th term of the sequence is $n^2 + 3n - 1$.

(ii) The sequence has n th term $an^2 + bn + c$.

Terms	-2	4	14	28	46
Differences		6	10	14	18
Second differences			4	4	4

So $a = 2$

Terms	-2	4	14	28	46
an^2	2	8	18	32	50
$bn + c$	-4	-4	-4	-4	-4

The values of $bn + c$ are all the same, so $b = 0$, and $c = -4$

The n th term of the sequence is $2n^2 - 4$.

(iii) The sequence has n th term $an^2 + bn + c$.

Terms	7	12	15	16	15
Differences		5	3	1	-1
Second differences			-2	-2	-2

So $a = -1$

Terms	7	12	15	16	15
an^2	-1	-4	-9	-16	-25
$bn + c$	8	16	24	32	40

The values of $bn + c$ go up by 8 each time, so $b = 8$, and $c = 0$

The n th term of the sequence is $-n^2 + 8n$.

$$7. (i) \quad n\text{th term} = \frac{2n+5}{4n-1}$$

$$1^{\text{st}} \text{ term} = \frac{2 \times 1 + 5}{4 \times 1 - 1} = \frac{2+5}{4-1} = \frac{7}{3}$$

$$5^{\text{th}} \text{ term} = \frac{2 \times 5 + 5}{4 \times 5 - 1} = \frac{10+5}{20-1} = \frac{15}{19}$$

$$100^{\text{th}} \text{ term} = \frac{2 \times 100 + 5}{4 \times 100 - 1} = \frac{200+5}{400-1} = \frac{205}{399}$$

As $n \rightarrow \infty$, $2n+5 \rightarrow 2n$, $4n-1 \rightarrow 4n$

$$\text{so } \frac{2n+5}{4n-1} \rightarrow \frac{2n}{4n} = \frac{1}{2}$$

The limit of the sequence is $\frac{1}{2}$.

$$(ii) \quad n\text{th term} = \frac{1-6n}{2n+3}$$

$$1^{\text{st}} \text{ term} = \frac{1-6 \times 1}{2 \times 1 + 3} = \frac{1-6}{2+3} = -\frac{5}{5} = -1$$

$$5^{\text{th}} \text{ term} = \frac{1-6 \times 5}{2 \times 5 + 3} = \frac{1-30}{10+3} = -\frac{29}{13}$$

$$100^{\text{th}} \text{ term} = \frac{1-6 \times 100}{2 \times 100 + 3} = \frac{1-600}{200+3} = -\frac{599}{203}$$

As $n \rightarrow \infty$, $1-6n \rightarrow -6n$, $2n+3 \rightarrow 2n$

$$\text{so } \frac{1-6n}{2n+3} \rightarrow \frac{-6n}{2n} = -3$$

8. Method 1

$$n^2 + 2n - 5 = 1000$$

$$n^2 + 2n = 1005$$

$$n^2 + 2n + 1 = 1006$$

$$(n+1)^2 = 1006$$

$n+1$ is a whole number but 1006 is not a square number so 1000 cannot be a term in the sequence.

Method 2

$$n^2 + 2n - 5 = 1000$$

$$n^2 + 2n = 1005$$

$$n(n+2) = 1005$$

n and $n+2$ are consecutive even or consecutive odd numbers. To multiply to make 1005, they must both be odd.

$$1005 = 3 \times 5 \times 67$$

There are no consecutive odd numbers that multiply to make 1005 so 1000 cannot be a term in the sequence.

Method 3

$$n^2 + 2n - 5 = 1000$$

$$n^2 + 2n - 1005 = 0$$

Solve the quadratic equation.

$$n = 30.72 \text{ or } -32.72$$

n is not an integer so 1000 is not a term in the sequence.

9. (a) One possible sequence is $\frac{3n}{n+1}$

(b) One possible sequence is $4 - \frac{n}{n+1}$