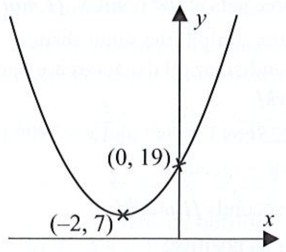
Set 2 Paper 1 — Pure Mathematics 1

1 a) $b^2 - 4ac = 12^2 - 4(3)(19) = 144 - 228 = -84$ ***[1 mark]***
The discriminant is negative, so the graph does not intersect the x-axis. ***[1 mark]***
[2 marks available in total — as above]

b) $3x^2 + 12x + 19 = 3(x^2 + 4x) + 19$ ***[1 mark]***
$= 3((x+2)^2 - 4) + 19$ ***[1 mark]*** $= 3(x+2)^2 + 7$ ***[1 mark]***
[3 marks available in total — as above]

c)



[3 marks available — 1 mark for correct shape with minimum in the correct quadrant, 1 mark for labelling the stationary point (–2, 7), 1 mark for labelling the y-intercept (0, 19)]

d) A translation 2 units right and 7 units down or by vector $\begin{pmatrix}2\\-7\end{pmatrix}$ ***[1 mark]***, then a vertical stretch by scale factor $\frac{1}{3}$ ***[1 mark]***.
[2 marks available in total — as above]
Alternatively, a vertical stretch by scale factor $\frac{1}{3}$ followed by a translation 2 units right and $\frac{7}{3}$ units down.

2 If w is the width then the length is $2w$.
The perimeter can be up to 40 m,
so $w + 2w + w + 2w \le 40 \Rightarrow 6w \le 40 \Rightarrow w \le \frac{20}{3}$ ***[1 mark]***.
The area needs to be at least 60 m²,
so $w \times 2w \ge 60 \Rightarrow 2w^2 \ge 60 \Rightarrow w^2 \ge 30$
$\Rightarrow w \le -\sqrt{30}$ or $w \ge \sqrt{30}$ ***[1 mark]***
Ignoring negative values (as the width can't be negative) gives $\sqrt{30} \le w \le \frac{20}{3}$. So the difference between the maximum and minimum width is:
$\frac{20}{3} - \sqrt{30}$ ***[1 mark]*** $= \frac{20 - 3\sqrt{30}}{3} = \frac{20 - \sqrt{270}}{3}$ m ***[1 mark]***
[4 marks available in total — as above]

3 a) $\frac{dy}{dx} = 2e^{2x}$ ***[1 mark]***
a is directly proportional to $b \Rightarrow a = kb$ for some constant k.
Here, $y = \frac{1}{2}\frac{dy}{dx}$, so y is directly proportional to $\frac{dy}{dx}$. ***[1 mark]***
[2 marks available in total — as above]

b) At $(2, e^4)$, $\frac{dy}{dx} = 2e^4$. Using $y - y_1 = m(x - x_1)$ gives
$y - e^4 = 2e^4(x - 2)$ ***[1 mark]*** $\Rightarrow y = 2e^4x - 3e^4$
The line crosses the x-axis at $y = 0$, so $0 = 2e^4x - 3e^4$
$\Rightarrow x = \frac{3}{2}$ so the coordinates are $(\frac{3}{2}, 0)$ ***[1 mark]***
[2 marks available in total — as above]

4 a) Freya is incorrect ***[1 mark]***. Although the value of $\sin x$ does repeat every 2π, the value of x^2 does not, so she needs to show it for all values of x. ***[1 mark]***
[2 marks available in total — as above]

b) For all values of x, $3\sin x \ge -3$ so $3\sin x + 3 \ge 0$
When $x \ne 0$, $x^2 > 0$ for all values of x.
So $3\sin x + x^2 + 3 > 0$ when $x \ne 0$
When $x = 0$, $3\sin 0 + 0^2 + 3 = 3$ so
$3\sin x + x^2 + 3 > 0$ for all values of x.
[3 marks available — 1 mark for stating that $3\sin x \ge -3$, 1 mark for arguing that x^2 is positive for non-zero values, 1 mark for a fully correct proof]

5 $f(-2) = 0 \Rightarrow (-2)^3 - b(-2)^2 + 2(-2) + 40 = 0$ ***[1 mark]***
$\Rightarrow -8 - 4b - 4 + 40 = 0 \Rightarrow 4b = 28 \Rightarrow b = 7$ ***[1 mark]***
It cuts the x-axis at -2 so $(x + 2)$ is a factor ***[1 mark]***

Dividing the cubic expression by $(x + 2)$ gives:

$$\begin{array}{r l} & x^2 - 9x + 20 \\ x+2 \,\big) & x^3 - 7x^2 + 2x + 40 \\ & \underline{x^3 + 2x^2} \\ & \quad -9x^2 + 2x \\ & \quad \underline{-9x^2 - 18x} \\ & \qquad 20x + 40 \\ & \qquad \underline{20x + 40} \\ & \qquad\quad 0 \end{array}$$

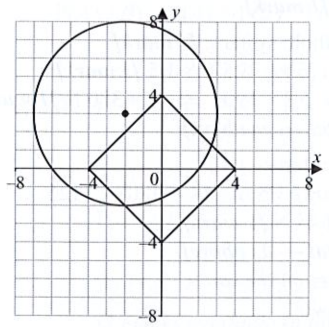
[1 mark for a correct method]

$f(x) = (x + 2)(x^2 - 9x + 20)$ ***[1 mark]*** $= (x + 2)(x - 4)(x - 5)$ ***[1 mark]***
[6 marks available in total — as above]
You could have used an alternative method to take a factor of (x + 2) out of the expression.

6 a) The formula for a geometric sequence is $u_n = ar^{n-1}$
The ball is dropped from 6 m so $a = 6$.
After the first bounce, $u_2 = 5.52$ m so:
$u_2 = a \times r^1 \Rightarrow 5.52 = 6 \times r \Rightarrow r = 0.92 \Rightarrow u_n = 6 \times 0.92^{n-1}$
When the maximum height for a bounce is less than 1 m:
$u_n < 1 \Rightarrow 6 \times 0.92^{n-1} < 1 \Rightarrow 0.92^{n-1} < \frac{1}{6}$
$\Rightarrow \ln 0.92^{n-1} < \ln\frac{1}{6} \Rightarrow (n-1)\ln 0.92 < \ln\frac{1}{6}$
$\Rightarrow n - 1 > \frac{\ln\frac{1}{6}}{\ln 0.92} \Rightarrow n - 1 > 21.4... \Rightarrow n > 22.4...$
So a maximum height of less than 1 metre is first achieved when $n = 23$, which is the 23rd maximum height, so after the 22nd bounce.
[5 marks available — 1 mark for finding the correct value of r, 1 mark for setting up the formula for a geometric sequence, 1 mark for using logs to solve the inequality, 1 mark for solving the inequality correctly, 1 mark for interpreting the answer in the context of the question]

b) $S_\infty = \frac{a}{1-r} = \frac{6}{1 - 0.92} = \frac{6}{0.08} = 75$ m ***[1 mark]***
After each bounce the ball goes up and down, so you need to double S_∞, but it only travels the initial height (6 m) once, so:
Total distance travelled = (2 × 75) – 6 = 144 m ***[1 mark]***
[2 marks available in total — as above]

7



From the diagram you can see that the circle intersects the square twice, and that one of the points of intersection is (–2, –2) ***[1 mark]***.
To find the other point of intersection, in the top-right quadrant, solve the equations simultaneously.
The circle has equation $(x + 2)^2 + (y - 3)^2 = 5^2$ ***[1 mark]***
The equation of the line which makes up the square where $x > 0$ and $y > 0$ has gradient -1 and y-intercept 4, so the equation is $y = -x + 4$ ***[1 mark]***
Substitute $y = -x + 4$ into $(x + 2)^2 + (y - 3)^2 = 25$ to give
$(x + 2)^2 + ((-x + 4) - 3)^2 = 25$ ***[1 mark]***
$\Rightarrow (x + 2)^2 + (-x + 1)^2 = 25 \Rightarrow (x^2 + 4x + 4) + (x^2 - 2x + 1) = 25$
$\Rightarrow 2x^2 + 2x - 20 = 0 \Rightarrow x^2 + x - 10 = 0$ ***[1 mark]***
$\Rightarrow x = \frac{-1 \pm \sqrt{1^2 + 40}}{2} \Rightarrow x = \frac{-1 \pm \sqrt{41}}{2}$
Ignore the negative root since $x > 0$.
$\Rightarrow x = \frac{-1 + \sqrt{41}}{2} = 2.701...$ ***[1 mark]***
At $x = 2.701...$, $y = -x + 4 = -2.701... + 4 = 1.298...$
So the coordinates of the other point of intersection are (2.70, 1.30) (2 d.p.) ***[1 mark]***
[7 marks available in total — as above]

8 a) $f^{-1}(x) = \frac{2x - 5}{x} \Rightarrow x = \frac{2y - 5}{y} \Rightarrow yx = 2y - 5$
$\Rightarrow yx - 2y = -5 \Rightarrow y(x - 2) = -5 \Rightarrow y = \frac{-5}{x - 2}$
$\Rightarrow f(x) = \frac{-5}{x - 2}$ or $\frac{5}{2 - x}$
So $fg(x) = \frac{-5}{\sqrt{2x - k} - 2}$ or $\frac{5}{2 - \sqrt{2x - k}}$
$fg(x)$ is undefined when $2x - k < 0 \Rightarrow x < \frac{k}{2}$
or $\sqrt{2x - k} - 2 = 0 \Rightarrow 2x - k = 4 \Rightarrow x = 2 + \frac{k}{2}$
So $fg(x)$ has domain $x \ge \frac{k}{2}$ and $x \ne 2 + \frac{k}{2}$
[3 marks available — 1 mark for finding f(x), 1 mark for finding fg(x), 1 mark for identifying a suitable domain]

b) $g(10) = \sqrt{20 - k}$ and $gg(10) = \sqrt{2\sqrt{20 - k} - k} = 2$ ***[1 mark]***
$\Rightarrow 2\sqrt{20 - k} - k = 4 \Rightarrow \sqrt{20 - k} = \frac{4 + k}{2}$
$\Rightarrow 20 - k = \left(\frac{4 + k}{2}\right)^2 \Rightarrow 20 - k = \frac{16 + 8k + k^2}{4}$
$\Rightarrow 80 - 4k = 16 + 8k + k^2$ ***[1 mark]***
$\Rightarrow k^2 + 12k - 64 = 0 \Rightarrow (k + 16)(k - 4) = 0$
$\Rightarrow k = 4$ ***[1 mark]*** (ignore $k = -16$ as k is positive)
[3 marks available in total — as above]

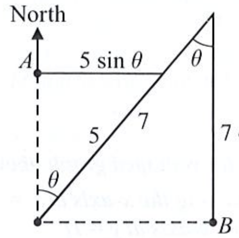
9 a) $6y\frac{dy}{dx} - 4\frac{dy}{dx} = -6x^2$
$\Rightarrow \frac{dy}{dx}(6y - 4) = -6x^2 \Rightarrow \frac{dy}{dx} = \frac{-3x^2}{3y - 2}$ or $\frac{3x^2}{2 - 3y}$
[3 marks available — 1 mark for differentiating $3y^2$ correctly, 1 mark for differentiating the other terms correctly, 1 mark for the correct answer]

b) At stationary points, $\frac{dy}{dx} = 0 \Rightarrow \frac{-3x^2}{3y - 2} = 0 \Rightarrow x = 0$ ***[1 mark]***
At $x = 0$, $3y^2 - 4y = 4 \Rightarrow 3y^2 - 4y - 4 = 0$ ***[1 mark]***
$\Rightarrow (3y + 2)(y - 2) = 0 \Rightarrow y = -\frac{2}{3}$ or $y = 2$
So the distance between the stationary points is:
$2 - \left(-\frac{2}{3}\right) = 2\frac{2}{3}$ or $\frac{8}{3}$ ***[1 mark]***
[3 marks available in total — as above]

10 $x = \frac{2}{3}\sin\theta \Rightarrow \frac{dx}{d\theta} = \frac{2}{3}\cos\theta$ ***[1 mark]*** $\Rightarrow dx = \frac{2}{3}\cos\theta\, d\theta$
So $\int \frac{1}{\sqrt{4 - 9x^2}}\,dx = \int \frac{1}{\sqrt{4 - 9 \times \frac{4}{9}\sin^2\theta}}\frac{2}{3}\cos\theta\, d\theta$ ***[1 mark]***
$= \frac{2}{3}\int \frac{\cos\theta}{\sqrt{4 - 4\sin^2\theta}}\,d\theta = \frac{2}{3}\int \frac{\cos\theta}{\sqrt{4}\sqrt{1 - \sin^2\theta}}\,d\theta = \frac{2}{6}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}}\,d\theta$
$= \frac{1}{3}\int \frac{\cos\theta}{\cos\theta}\,d\theta = \frac{1}{3}\int 1\,d\theta$ ***[1 mark]*** $= \frac{\theta}{3} + C$ ***[1 mark]***
Rearranging $x = \frac{2}{3}\sin\theta$ gives $\theta = \arcsin\frac{3}{2}x$,
so $\int \frac{1}{\sqrt{4 - 9x^2}}\,dx = \frac{1}{3}\arcsin\frac{3}{2}x + C$ ***[1 mark]***
[5 marks available in total — as above]

11 a) $R\sin(\theta + \alpha) = 5\sin\theta + 7\cos\theta$
$\Rightarrow R\sin\theta\cos\alpha + R\cos\theta\sin\alpha = 5\sin\theta + 7\cos\theta$
$\Rightarrow R\cos\alpha = 5$ and $R\sin\alpha = 7$
$\frac{R\sin\alpha}{R\cos\alpha} = \frac{7}{5} \Rightarrow \tan\alpha = 1.4$ ***[1 mark]***
$\Rightarrow \alpha = 54.46...° = 54°$ (nearest whole degree) ***[1 mark]***
$(R\sin\alpha)^2 + (R\cos\alpha)^2 = 5^2 + 7^2 \Rightarrow R^2(\sin^2\alpha + \cos^2\alpha) = 25 + 49$
$\Rightarrow R^2 = 74 \Rightarrow R = \sqrt{74}$ ***[1 mark]***
[3 marks available in total — as above]

b)



Boat A sails a distance of $5 + 5\sin\theta$
Boat B sails a distance of $7 + 7\cos\theta$ ***[1 mark for both]***
$12 + 5\sin\theta + 7\cos\theta = 18 \Rightarrow 5\sin\theta + 7\cos\theta = 6$
$\sqrt{74}\sin(\theta + 54.46...°) = 6$ ***[1 mark]***
$\Rightarrow \sin(\theta + 54.46...°) = \frac{6}{\sqrt{74}}$
$\Rightarrow \theta + 54.46...° = 44.22...°, 135.77...°$
$\Rightarrow \theta = -10.24...°, 81.31...°$ ***[1 mark]***
You know that $0° < \theta < 90° \Rightarrow \theta = 81.31...°$, so the boats sail on a bearing of 081° (nearest whole degree) ***[1 mark]***.
[4 marks available in total — as above]

12 a) After t years, where A is the original sum and P is the value of the investment at that time: $P = A\left(\frac{100 + r}{100}\right)^t$
When the investment has doubled, $P = 2A$ and $t = T$;
substituting in gives: $2A = A\left(\frac{100 + r}{100}\right)^T$
$\Rightarrow 2 = \left(\frac{100 + r}{100}\right)^T \Rightarrow \log 2 = \log\left(\frac{100 + r}{100}\right)^T$
$\Rightarrow \log 2 = T\log\left(\frac{100 + r}{100}\right)$
$\Rightarrow T = \frac{\log 2}{\log\left(\frac{100 + r}{100}\right)} = \log_{\left(\frac{100+r}{100}\right)} 2$ or $\log_{(1 + 0.01r)} 2$
[3 marks available — 1 mark for a correct equation relating T and r, 1 mark for rearranging and taking logs, 1 mark for the correct answer]

b) $15 = \log_{\left(\frac{100+r}{100}\right)} 2 \Rightarrow \left(\frac{100 + r}{100}\right)^{15} = 2$ ***[1 mark]***
$\sqrt[15]{2} = \frac{100 + r}{100} \Rightarrow 1.0472... = \frac{100 + r}{100}$
$\Rightarrow 100 + r = 104.729...$
$\Rightarrow r = 4.729...\% = 4.73\%$ (2 d.p.) ***[1 mark]***
[2 marks available in total — as above]

c) From a), compound interest will double when $t = \log_{\left(\frac{100+q}{100}\right)} 2$.
The simple interest account doubles after $2t$ years:
If the initial investment was A, then
$2A = A \times \left(\frac{100 + 2pt}{100}\right) \Rightarrow \left(\frac{100 + 2pt}{100}\right) = 2$ ***[1 mark]***
$\Rightarrow 200 = 100 + 2pt \Rightarrow 2t = \frac{100}{p} \Rightarrow t = \frac{50}{p}$ ***[1 mark]***
Substituting in $t = \log_{\left(\frac{100+q}{100}\right)} 2$ gives:
$\log_{\left(\frac{100+q}{100}\right)} 2 = \frac{50}{p} \Rightarrow 2 = \left(\frac{100 + q}{100}\right)^{\frac{50}{p}}$ ***[1 mark]***
$\Rightarrow 2^{\frac{p}{50}} = \frac{100 + q}{100} \Rightarrow q = 100 \times 2^{\frac{p}{50}} - 100$
$\Rightarrow q = 100(2^{\frac{p}{50}} - 1)$ as required ***[1 mark]***
[4 marks available in total — as above]

13 a) Use the quotient rule with:
$u = x^2\cos x \Rightarrow \frac{du}{dx} = (2x\cos x) + (x^2(-\sin x))$
$= 2x\cos x - x^2\sin x$ ***[1 mark]***
$v = 3\sin x \Rightarrow \frac{dv}{dx} = 3\cos x$ ***[1 mark]***
$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$= \frac{(3\sin x)(2x\cos x - x^2\sin x) - (x^2\cos x)(3\cos x)}{(3\sin x)^2}$ ***[1 mark]***
$= \frac{6x\sin x\cos x - 3x^2\sin^2 x - 3x^2\cos^2 x}{9\sin^2 x}$
$= \frac{x(2\sin x\cos x - x(\sin^2 x + \cos^2 x))}{3\sin^2 x}$ ***[1 mark]***
Using $2\sin x\cos x = \sin 2x$ and $\sin^2 x + \cos^2 x = 1$:
$= \frac{x(\sin 2x - x)}{3\sin^2 x}$ as required. ***[1 mark]***
[5 marks available in total — as above]
Alternatively, you could write $\frac{x^2\cos x}{3\sin x} = \frac{1}{3}x^2\cot x$ and use the product rule to differentiate.

b) At stationary points, $\frac{dy}{dx} = 0$. If $a = 0.95$ to 2 decimal places, then $0.945 \le a < 0.955$. ***[1 mark]***
When $x = 0.945$, $\frac{dy}{dx} = 0.00215...$ (positive)
When $x = 0.955$, $\frac{dy}{dx} = -0.00572...$ (negative)
There is a change of sign between 0.945 and 0.955, so $a = 0.95$ to 2 decimal places. ***[1 mark]***
[2 marks available in total — as above]

c) When $x = 0.95$, $\frac{dy}{dx} = -0.00177...$ (negative) so the root is between 0.945 and 0.95, i.e. 0.95 is an overestimate. ***[1 mark]***

14 a) $\frac{dV}{dt} = \frac{2}{Vt} \Rightarrow \int \frac{V}{2} dV = \int \frac{1}{t} dt$
$\Rightarrow \frac{V^2}{4} = \ln|t| + C = \ln t + C$ as $t > 0$
When $t = 1$, there are 4000 views so $V = 4$.
So $\frac{4^2}{4} = \ln 1 + C \Rightarrow C = 4 \Rightarrow \frac{V^2}{4} = \ln t + 4$
$\Rightarrow V^2 = 4(\ln t + 4) \Rightarrow V = 2\sqrt{\ln t + 4}$
At the end of day 7, $V = 2\sqrt{\ln 7 + 4} = 4.876...$
At the end of day 8, $V = 2\sqrt{\ln 8 + 4} = 4.931...$
So during day 8 it got $1000 \times (4.931... - 4.876...) = 54.457... = 54$ views.
[5 marks available — 1 mark for the correct method for integration, 1 mark for the correct integration, 1 mark for finding the value of C, 1 mark for a correct method to find the number of views on day 8, 1 mark for the correct answer]

b) E.g. The model is undefined when $\ln t < -4$ (i.e. $t < 0.0183...$). Theo could improve the model by giving a different equation for V during this time (e.g. $V = 0$ for $t < 0.0183...$).
[2 marks available — 1 mark for a limitation of the model, 1 mark for a suitable improvement to address the given limitation]
There are other limitations that you could mention — for example, the model suggests that the views will continue to increase forever, so an upper limit on t might be needed.

15 a) $\cos 2x = \cos^2 x - \sin^2 x$
$= (1 - \sin^2 x) - \sin^2 x = 1 - 2\sin^2 x$ ***[1 mark]***
$\Rightarrow \sin^2 x = \frac{1 - \cos 2x}{2}$
$\Rightarrow \sin^2 5x = \frac{1 - \cos 10x}{2} = \frac{1}{2}(1 - \cos 10x)$ ***[1 mark]***
$\int \sin^2 5x \, dx = \frac{1}{2}\int 1 - \cos 10x \, dx$
$= \frac{1}{2}\left(x - \frac{\sin 10x}{10}\right) + C$ or $\frac{x}{2} - \frac{\sin 10x}{20} + C$
[1 mark for $\frac{x}{2}$, 1 mark for $-\frac{\sin 10x}{20}$]
[4 marks available in total — as above]

b) Using integration by parts: $u = x$, $\frac{dv}{dx} = \sin^2 5x$
$\frac{du}{dx} = 1$ and $v = \frac{x}{2} - \frac{\sin 10x}{20}$
$\int_0^{\frac{2\pi}{5}} x\sin^2 5x \, dx = \left[x\left(\frac{x}{2} - \frac{\sin 10x}{20}\right)\right]_0^{\frac{2\pi}{5}} - \int_0^{\frac{2\pi}{5}} 1\left(\frac{x}{2} - \frac{\sin 10x}{20}\right) dx$
$= \left[\frac{x^2}{2} - \frac{x\sin 10x}{20} - \left(\frac{x^2}{4} + \frac{\cos 10x}{200}\right)\right]_0^{\frac{2\pi}{5}}$
$= \left[\frac{x^2}{4} - \frac{x\sin 10x}{20} - \frac{\cos 10x}{200}\right]_0^{\frac{2\pi}{5}}$
$= \left[\frac{\left(\frac{2\pi}{5}\right)^2}{4} - \frac{\frac{2\pi}{5}\sin\frac{10 \times 2\pi}{5}}{20} - \frac{\cos\frac{10 \times 2\pi}{5}}{200}\right] - \left[\frac{0^2}{4} - \frac{0\sin(10 \times 0)}{20} - \frac{\cos(10 \times 0)}{200}\right]$
$= \left[\frac{\pi^2}{25} - 0 - \frac{1}{200}\right] - \left[0 - 0 - \frac{1}{200}\right] = \frac{\pi^2}{25}$
[5 marks available — 1 mark for attempting to use integration by parts, 1 mark for applying the integration by parts formula correctly, 1 mark for the correct integral, 1 mark for substituting in limits of the integral, 1 mark for the correct answer]

Set 2 Paper 2 — Pure Mathematics 2

1 a) $\overrightarrow{AB} = \begin{pmatrix}-4\\1\\1\end{pmatrix} - \begin{pmatrix}4\\1\\5\end{pmatrix} = \begin{pmatrix}-8\\0\\-4\end{pmatrix}$ ***[1 mark]***
$|\overrightarrow{AB}| = \sqrt{(-8)^2 + 0^2 + (-4)^2}$ ***[1 mark]*** $= \sqrt{80} = 4\sqrt{5}$ ***[1 mark]***
[3 marks available in total — as above]

b) $\overrightarrow{AC} = 3 \times \begin{pmatrix}-8\\0\\-4\end{pmatrix} = \begin{pmatrix}-24\\0\\-12\end{pmatrix}$ ***[1 mark]***
$\overrightarrow{OC} = \begin{pmatrix}4\\1\\5\end{pmatrix} + \begin{pmatrix}-24\\0\\-12\end{pmatrix} = \begin{pmatrix}-20\\1\\-7\end{pmatrix}$ ***[1 mark]***
[2 marks available in total — as above]

2 $64^a \times \left(\frac{1}{16}\right)^b \div \sqrt[c]{32} = (2^6)^a \times (2^{-4})^b \div (2^5)^{\frac{1}{c}}$
$= 2^{6a} \times 2^{-4b} \div 2^{\frac{5}{c}} = 2^{6a - 4b - \frac{5}{c}}$
So $d = 6a - 4b - \frac{5}{c}$
[3 marks available — 1 mark for correctly rewriting one term as a power of 2, 1 mark for expressing all terms as powers of 2, 1 mark for the correct answer]

3 a) £5069 is the amount of money that the farmer would make if she sold her maize crop when $t = 0$ (on 1st July). ***[1 mark]***

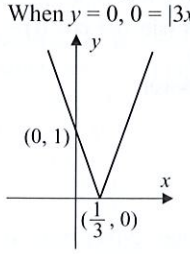
b) $\frac{dP}{dt} = -2t + 66$ ***[1 mark]*** At stationary points, $\frac{dP}{dt} = 0$
$\Rightarrow 0 = -2t + 66 \Rightarrow t = 33$ ***[1 mark]***
Since P is a quadratic with a negative coefficient of t^2, the turning point is a maximum ***[1 mark]***, so the optimum selling date is 33 days after 1st July, which is 3rd August. ***[1 mark]***
[4 marks available in total — as above]
You could also have justified your answer by finding the second derivative and showing it's negative at t = 33, so it's a maximum.

c) Substituting $t = 33$ into the equation for P gives $P = -(33^2) + (66 \times 33) + 5069 = £6158$. ***[1 mark]***

d) For sufficiently large t, e.g. $t = 200$, P is negative which doesn't make sense as P is the amount she sells the crop for. The value of t could be restricted in order to improve the model, e.g. by making $0 \le t \le 111$ as $t = 111$ is the last day where P is positive.
[2 marks available — 1 mark for any suitable limitation, 1 mark for a sensible suggestion for how it can be improved]

4 $4\cos x - 11 = \frac{\sin^2 x - 3}{\cos x}$
$\Rightarrow 4\cos^2 x - 11\cos x = \sin^2 x - 3$ ***[1 mark]***
$\Rightarrow 4\cos^2 x - 11\cos x = (1 - \cos^2 x) - 3$ ***[1 mark]***
$\Rightarrow 5\cos^2 x - 11\cos x + 2 = 0$ ***[1 mark]***
$\Rightarrow (5\cos x - 1)(\cos x - 2) = 0$ ***[1 mark]***
$\Rightarrow \cos x = 0.2$ ***[1 mark]*** ($\cos x \ne 2$ as $-1 \le \cos x \le 1$)
$\Rightarrow x = \cos^{-1}(0.2) = 78.46...° = 78.5°$ (1 d.p.)
and $x = 360° - 78.46...° = 281.53...° = 281.5°$ (1 d.p.)
[1 mark for both]
[6 marks available in total — as above]

5 a) When $x = 0$, $y = |3(0) - 1| = 1$
When $y = 0$, $0 = |3x - 1| \Rightarrow 3x = 1 \Rightarrow x = \frac{1}{3}$



[3 marks available — 1 mark for v-shaped graph above the x-axis, 1 mark for a graph touching the x-axis at $x = \frac{1}{3}$, 1 mark for a graph crossing the y-axis at y = 1]
You could also start with the graph y = 3x − 1, which has gradient 3 and y-intercept −1. Then any point below the x-axis gets reflected in the x-axis to produce the graph of y = |3x − 1|.

b) First, solve the equation $|3x - 1| = 2x + 5$:
When $3x - 1 \ge 0$, the equation becomes
$3x - 1 = 2x + 5 \Rightarrow x = 6$ ***[1 mark]***
When $3x - 1 < 0$, the equation becomes
$-3x + 1 = 2x + 5 \Rightarrow -5x = 4 \Rightarrow x = -0.8$ ***[1 mark]***
Since it is given that the equation has 2 solutions, they must be at $x = 6$ and $x = -0.8$
As the graph of $y = |3x - 1|$ is v-shaped, the inequality $|3x - 1| \le 2x + 5$ will be satisfied between the two solutions, i.e. when $-0.8 \le x \le 6$ ***[1 mark]***
[3 marks available in total — as above]

6 $S_{20} = 1390$ and $S_{30} = 3135$
Using the formula $S_n = \frac{1}{2}n[2a + (n - 1)d]$:
(1): For S_{20}, $1390 = 10(2a + 19d) = 20a + 190d$ ***[1 mark]***
(2): For S_{30}, $3135 = 15(2a + 29d) = 30a + 435d$ ***[1 mark]***
(1) × 3: $4170 = 60a + 570d$
(2) × 2: $6270 = 60a + 870d$
Now subtract one equation from the other to give:
$2100 = 300d \Rightarrow d = 7$ ***[1 mark]*** Substituting back into (1) gives:
$1390 = 20a + 190 \times 7 \Rightarrow 60 = 20a \Rightarrow a = 3$ ***[1 mark]***
Now if $a = 3$ and $d = 7$, $u_{10} = 3 + (9 \times 7) = 66$ prizes ***[1 mark]***
[5 marks available in total — as above]

7 a) Rearranging $x = 3t + 1$ to make t the subject gives $\frac{x - 1}{3} = t$.
Substituting this into $y = (t + 3)^3 - 5$ gives
$y = \left(\frac{x-1}{3} + 3\right)^3 - 5 = \left(\frac{x-1}{3} + \frac{9}{3}\right)^3 - 5 = \left(\frac{x+8}{3}\right)^3 - 5$
[2 marks available — 1 mark for rearranging to make t the subject, 1 mark for substituting and rearranging to give the required result]

b) Caleb hasn't changed the limits of the integration — he needs to change them to be in terms of t ***[1 mark]***.
He also hasn't multiplied by $\frac{dx}{dt}$ ***[1 mark]***.
[2 marks available in total — as above]

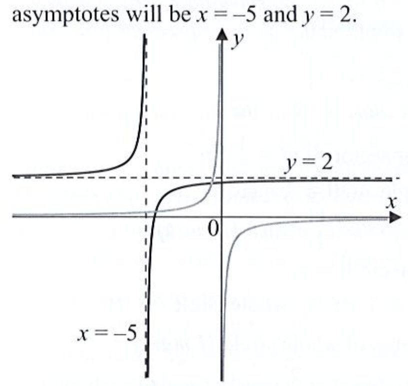
c) The limits of the integration become:
$x = 4 \Rightarrow 4 = 3t + 1 \Rightarrow t = 1$
$x = 10 \Rightarrow 10 = 3t + 1 \Rightarrow t = 3$
$\frac{dx}{dt} = 3$ and $y = (t + 3)^3 - 5$
So $\int_4^{10} y \, dx = 3\int_1^3 ((t + 3)^3 - 5) \, dt = 3\left[\frac{1}{4}(t + 3)^4 - 5t\right]_1^3$
$= 3\left[\frac{6^4}{4} - 15 - \left(\frac{4^4}{4} - 5\right)\right] = 750$
[3 marks available — 1 mark for correcting both of Caleb's errors (i.e. converting the limits and finding $\frac{dx}{dt}$), 1 mark for integrating correctly, 1 mark for substituting in the limits to obtain the correct final answer]

8 a) $y = at^b \Rightarrow \log_{10} y = \log_{10}(at^b) \Rightarrow \log_{10} y = \log_{10} a + \log_{10} t^b$
$\Rightarrow \log_{10} y = \log_{10} a + b\log_{10} t$
[2 marks available — 1 mark for taking logs of both sides, 1 mark for using laws of logs to simplify]

b) Use the graph to find the values of $\log_{10} a$ and b:
$\log_{10} a$ is the vertical intercept, which is 2.475 ***[1 mark]***.
$b = \text{gradient} = \frac{\text{change in } \log_{10} y}{\text{change in } \log_{10} t} = \frac{2.60 - 2.55}{0.25 - 0.15} = 0.5$ ***[1 mark]***
So the equation of the line of best fit is:
$\log_{10} y = 2.475 + 0.5\log_{10} t$
When $y = 1000$: $\log_{10} 1000 = 2.475 + 0.5\log_{10} t$ ***[1 mark]***
$\Rightarrow 0.5\log_{10} t = 3 - 2.475 \Rightarrow \log_{10} t = 1.05$
$\Rightarrow t = 11.220... = 11$ days (nearest whole day) ***[1 mark]***
[4 marks available in total — as above]
You could also work out the value of a and use the original equation ($y = 298.53... \times t^{0.5}$) to find the value of t.

c) E.g. The observed pattern might not continue — the rate of change could increase if the ants breed faster or more join the colony, or decrease if the breeding rate or number of ants joining the colony slows. / The colony might reach a certain size then remain at that size (e.g. due to restrictions on space or resources). / Once it gets to a certain size, ants might leave the colony to form a new one, so the number of ants could decrease. ***[1 mark for a sensible limitation linked to the number of ants in the colony]***

9 a) The graph of $y = \frac{-1}{x + 5} + 2$ will be a translation of $y = \frac{-1}{x}$ 5 units to the left and 2 units up. So the equations of the asymptotes will be $x = -5$ and $y = 2$.



[2 marks available — 1 mark for drawing the graph in the correct position, 1 mark for drawing and labelling the asymptotes correctly]

b) A stretch, scale factor 3, in the y-direction ***[1 mark]***
followed by a translation by the vector $\begin{pmatrix}0\\2\end{pmatrix}$ ***[1 mark]***.
[2 marks available in total — as above]
You could also have described it as a translation by the vector $\begin{pmatrix}0\\\frac{2}{3}\end{pmatrix}$ followed by a stretch in the y-direction by scale factor 3.

10 The curve has a root at −1 so $(x + 1)$ is a factor. ***[1 mark]***
$x^3 - 5x^2 + 2x + 8 = (x + 1)(x^2 - 6x + 8)$ ***[1 mark]***
$= (x + 1)(x - 4)(x - 2)$
So the x-value at point C must be $x = 2$ ***[1 mark]***
Shaded area $= \left(\frac{1}{2} \times 1 \times 6\right) + \int_1^2 (x^3 - 5x^2 + 2x + 8) \, dx$ ***[1 mark]***
$= 3 + \left[\frac{x^4}{4} - \frac{5x^3}{3} + x^2 + 8x\right]_1^2$ ***[1 mark]***
$= 3 + \left[\frac{16}{4} - \frac{40}{3} + 4 + 16\right] - \left[\frac{1}{4} - \frac{5}{3} + 1 + 8\right]$ ***[1 mark]***
$= 3 + \frac{32}{3} - \frac{91}{12} = \frac{73}{12}$ ***[1 mark]***
[7 marks available in total — as above]

11 a) Let $f(x) = x^3$ then
$f'(x) = \lim_{h \to 0}\left(\frac{(x + h)^3 - x^3}{h}\right)$ ***[1 mark]***
$= \lim_{h \to 0}\left(\frac{(x^2 + 2xh + h^2)(x + h) - x^3}{h}\right)$
$= \lim_{h \to 0}\left(\frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h}\right)$ ***[1 mark]***
$= \lim_{h \to 0}\left(\frac{3x^2h + 3xh^2 + h^3}{h}\right)$
$= \lim_{h \to 0}(3x^2 + 3xh + h^2)$ ***[1 mark]***
As $h \to 0$, $3x^2 + 3xh + h^2 \to 3x^2$, so $f'(x) = 3x^2$ ***[1 mark]***
[4 marks available in total — as above]

b) Let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x$
$y = \sin^3 x = u^3$ so $\frac{dy}{du} = 3u^2 = 3\sin^2 x$
Then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 3\cos x \sin^2 x$
Use the product rule where $u = 3\cos x$ and $v = \sin^2 x$:
$\Rightarrow \frac{du}{dx} = -3\sin x$ and $\frac{dv}{dx} = 2\sin x \cos x$